CZECH TECHNICAL UNIVERSITY IN PRAGUE

Faculty of Nuclear Sciences and Physical Engineering

Department of Mathematics

Study branch: Engineering Informatics

Specialization: Software Engineering and Mathematical Informatics

Abelian complexity of infinite words and Abelian return words

RESEARCH PROJECT

Author:	**Bc. Karel Břinda**
Supervisor:	**Prof. Ing. Edita Pelantová, CSc.**
Advisor:	**Ing. Ondřej Turek, Ph.D.**
Academic year:	**2011/12**

Prohlášení

Prohlašuji, že jsem předloženou práci vypracoval samostatně a že jsem uvedl veškerou použitou literaturu.

V Praze dne 1. září 2012 Karel Břinda

Acknowledgments

I would like to thank my advisors, Edita Pelantová and Ondřej Turek, for giving the ideas for this project, collaboration, and help. I wish to extend my thanks to the whole Theoretical Informatics Group (TIGR) at the FNSPE CTU in Prague for very congenial atmosphere.

This work was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS11/162/OHK4/3T/14.

Name: **Abelian complexity of infinite words and Abelian return words**
Author: Bc. Karel Břinda

Study branch: Engineering Informatics
Specialization: Software Engineering and Mathematical Informatics
Type of work: Research project

Supervisor: Prof. Ing. Edita Pelantová, CSc.
Deparment of Mathematics
Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague

Advisor: Ing. Ondřej Turek, Ph.D.
Laboratory of Physics
Kochi University of Technology

Abstract:
The main topics of interest in this text are c-balance and Abelian properties of infinite words. Namely we deal with balance function, Abelian complexity, Abelian returns, and semi-Abelian returns. Among summary of already known results, we prove the $(d-1)$-balance property of the d-bonacci word for $d \in \{2, \dots, 12\}$. We then use the obtained bounds on the balance function to estimate the Abelian complexity of this word, for which we also compute, using our computer programs, the Abelian complexity, the balance function, and we find the semi-Abelian returns.

Keywords: combinatorics on words, Abelian complexity, c-balance property, discrepancy, Abelian returns, semi-Abelian returns, d-bonacci word

Abstrakt:
Hlavním předmětem našeho zájmu jsou v tomto textu c-balancovanost a abelovské vlastnosti nekonečných slov. Jmenovitě se zabýváme balanční funkcí, abelovskou komplexitou, abelovskými a poloabelovskými návraty. Kromě shrnutí již známých výsledků dále dokazujeme $(d-1)$-balancovanost d-bonacciho slova pro $d \in \{2, \dots, 12\}$. S pomocí získaných mezí na balanční funkci odhadujeme abelovskou komplexitu tohoto slova, pro které dále s pomocí našich počítačových programů také napočíváme hodnoty abelovské komplexity, balanční funkce a hledáme polo-abelovské návraty.

Klíčová slova: kombinatorika na slovech, abelovská komplexita, c-balancovanost, diskrepance, abelovské návraty, poloabelovské návrat, d-bonacciho slovo

Contents

List of symbols

Symbol	Description
$\mathbb{Z}^+$	the positive integer numbers, $\{1, 2, \ldots\}$
$\mathbb{Z}_0^+$	the non-negative integer numbers, $\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	the integers, $\{\ldots - 2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$	the rational numbers
$\mathbb{C}$	the complex numbers
$\#M$	cardinality of the finite set M
$\binom{n}{k}$	binomial coefficient, $\frac{n!}{k!(n-k)!}$ if $n \geq k$, 0 otherwise
$\binom{n}{k}^{[l+1]}$	coefficient of x^k in the expansion of $(1 + x + \ldots + x^l)^n$
$\mathcal{A}$	alphabet
$\mathcal{A}^\star$	set of all finite words over $\mathcal{A}$
$\mathcal{A}^{\mathbb{Z}_0^+}$	set of all one-sided infinite words over $\mathcal{A}$
ε	empty word in $\mathcal{A}^\star$
$\boldsymbol{u}, \boldsymbol{v}$	infinite words over $\mathcal{A}$
u, v	finite words over $\mathcal{A}$
$\|u\|$	the number of letters in u, length of u
$\|u\|_a$	the number of occurrences of the letter a in u
$u_{[n,n+k)}$	factor $u_n u_{n+1} \cdots u_{n+k-1}$ of u
$\boldsymbol{u}_{[n,n+k)}$	factor $\boldsymbol{u}_n \boldsymbol{u}_{n+1} \cdots \boldsymbol{u}_{n+k-1}$ of $\boldsymbol{u}$
$\Psi(u)$	Parikh vector of u
$\mathcal{L}(\boldsymbol{u})$	set of factors of $\boldsymbol{u}$
$\mathcal{L}_n(\boldsymbol{u})$	set of factors of $\boldsymbol{u}$ of length n
Δ	end of a theorem, a proposition, a lemma, an observation, a corollary, a conjecture, a definition, or an example
$\square$	end of a proof

CHAPTER 1

Preliminaries

Alphabet is a finite non-empty set $\mathcal{A} = \{a_0, \dots, a_{d-1}\}$. Its elements are called **letters**. In this text, we will use the alphabet $\mathcal{A} = \{0, \dots, d-1\}$. **Finite word** u is a finite sequence over $\mathcal{A}$. The number of letters in the word u is called **length** of u and we denote it by $|u|$. A special case is a unique word of zero length called **empty word** and denoted by ε. Let us denote the number of occurrences of a letter $a \in \mathcal{A}$ in the word u by $|u|_a$.

$\mathcal{A}^*$ is the set of all finite words over $\mathcal{A}$ and $\mathcal{A}^+$ is the set of all finite words over $\mathcal{A}$ except ε. $\mathcal{A}^n$ is the subset of $\mathcal{A}^*$ containing all words of length n. $\mathcal{A}^{\mathbb{Z}_0^+}$ is the set of all **infinite words**, i.e., one-sided infinite sequences over $\mathcal{A}$.

We can equip the sets $\mathcal{A}^*$, $\mathcal{A}^+$ with a binary operation called **concatenation**

$$(v_1 v_2 \cdots v_m) \cdot (w_1 w_2 \cdots w_n) = v_1 v_2 \cdots v_m w_1 w_2 \cdots w_n$$

in order to get a monoid $(\mathcal{A}^*, \cdot)$ with the neutral element ε and a semigroup $(\mathcal{A}^+, \cdot)$ respectively.

The word u is **factor** of a word w if $w = vuv'$ for some $v \in \mathcal{A}^*, v' \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{Z}^+}$. In the case $v = \varepsilon$, we say that u is **prefix** of w. Similarly, if $v' = \varepsilon$, then v is **suffix** of w. To make the formulae in this text shorter, we denote $u_n u_{n+1} \cdots u_{n+k-1}$ by $u_{[n,n+k)}$.

The set of all factors of an infinite word $\boldsymbol{u}$ is called **language** and we denote it by $\mathcal{L}(\boldsymbol{u})$. Its subset containing just all factors of $\boldsymbol{u}$ of length n is denoted by $\mathcal{L}_n(\boldsymbol{u})$.

Let $\boldsymbol{u}$ be an infinite word and let w be its non-empty factor. The set of all letters $a \in \mathcal{A}$ such that $wa \in \mathcal{L}(\boldsymbol{u})$ is called **right extension** of w in $\boldsymbol{u}$. It is denoted by $Rext(w)$. If $\#Rext(w) > 1$, the word w is said to be **right special factor**. Similarly, we define **left extension** and **left special factor**. A factor which is left special and right special is called **bispecial factor**.

If every factor of an infinite word $\boldsymbol{u}$ occurs at least twice (which implies that it occurs infinitely many times), $\boldsymbol{u}$ is said to be **recurrent**. A word v is called **return word** of $w \in \mathcal{L}(\boldsymbol{u})$ if the following conditions are satisfied:

- $vw \in \mathcal{L}(\boldsymbol{u})$;
- w is a prefix of vw;
- w occurs in vw exactly twice.

The factor vw is said to be **complete return word** of w. Let us denote the set of all return words of w in $\boldsymbol{u}$ by $Ret(w)$.

A recurrent word $\boldsymbol{u}$ satisfying that for all w in $\mathcal{L}(\boldsymbol{u})$ it holds $\#Ret(w) < +\infty$ is called **uniformly recurrent**. If there exists $C > 0$ such that for all w in $\mathcal{L}(\boldsymbol{u})$ and all v in $Ret(w)$, it is satisfied $|v| < C|w|$; we call the word $\boldsymbol{u}$ **linearly recurrent**.

Let $u = u_0 u_1 \cdots u_{n-1}$ be a finite word. We define its **mirror image** as $\bar{u} = u_{n-1} \cdots u_1 u_0$. Words satisfying $u = \bar{u}$ are called **palindromes**.

An infinite word $\boldsymbol{u}$ is **eventually periodic** if $\boldsymbol{u} = vw^\omega$ for some $v, w \in \mathcal{A}^*$, where w^ω means repetition of w infinitely many times. In the special case $v = \varepsilon$, $\boldsymbol{u}$ is said to be **periodic**. Words which are not eventually periodic are **aperiodic**.

An infinite word $\boldsymbol{u}$ has **vector of frequencies** $\mu = (\mu_0, \ldots, \mu_{d-1})$ if there exist limits

$$\mu_a = \lim_{N \to +\infty} \frac{|\boldsymbol{u}_{[0,N)}|_a}{N}$$

for all letters $a \in \mathcal{A}$.

The set $\mathcal{A}^{\mathbb{Z}^+}$ can be equipped with a metric defined as

$$\rho(\boldsymbol{u}, \boldsymbol{v}) = \begin{cases} 2^{-\min\{m \in \mathbb{Z}_0^+ \mid \boldsymbol{u}_m \neq \boldsymbol{v}_m\}} & \text{for } \boldsymbol{u} \neq \boldsymbol{v}; \\ 0 & \text{for } \boldsymbol{u} = \boldsymbol{v}. \end{cases} \tag{1.1}$$

The metric ρ induces **product topology**. The set $\mathcal{A}^{\mathbb{Z}^+}$ is a compact topological space.

Among several ways of how to create an infinite word, one of the most common ones is to generate it by a special kind of morphisms on $\mathcal{A}^*$ – using the so-called substitution.

Definition 1.1. **Substitution** is a morphism $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ satisfying:

- there exists a letter $a \in \mathcal{A}$ such that a is a prefix of $\varphi(a)$;
- for all letters $b \in \mathcal{A}$, it holds $\lim_{n \to +\infty} |\varphi^n(b)| = +\infty$.

The action of the morphism φ can be naturally extended to an infinite word $\boldsymbol{u}$ by $\varphi(\boldsymbol{u}_0 \boldsymbol{u}_1 \boldsymbol{u}_2 \cdots) = \varphi(\boldsymbol{u}_0)\varphi(\boldsymbol{u}_1)\varphi(\boldsymbol{u}_2) \cdots$. An infinite word $\boldsymbol{u}$ is a fixed point of the morphism φ if $\boldsymbol{u} = \varphi(\boldsymbol{u})$. $\triangle$

We can mention a property which directly follows from the definition.

Lemma 1.2. Every substitution has a **fixed point**

$$\boldsymbol{u} = \lim_{n \to +\infty} \varphi^n(a),$$

where a is from Definition 1.1. $\triangle$

The class of all substitutions is very wide and hard to treat in full generality. Therefore, we consider an extensively studied subclass of primitive substitutions in this text.

Definition 1.3. A substitution φ is **primitive** if there exists $k \in \mathbb{Z}^+$ such that for all $a, b \in \mathcal{A}$, it is satisfied that $\varphi^k(a)$ contains b. $\triangle$

Proposition 1.4. Fixed points of primitive substitutions are linearly recurrent. $\triangle$

Now we can introduce a matrix which contains information about images of individual letters of the alphabet. Note that many properties of fixed points of substitutions are readable just from this matrix.

Definition 1.5. Let φ be a morphism over the alphabet $\mathcal{A} = \{0, 1, \dots, d-1\}$. The matrix

$$M_\varphi = \begin{pmatrix} |\varphi(0)|_0 & \cdots & |\varphi(d-1)|_0 \\ \vdots & \ddots & \vdots \\ |\varphi(0)|_{d-1} & \cdots & |\varphi(d-1)|_{d-1} \end{pmatrix}$$

is said to be **incidence matrix** of φ. Δ

It follows immediately from the definition of M_φ that for any word u over $\mathcal{A} = \{0, 1, \dots, d-1\}$

$$\begin{pmatrix} |\varphi(u)|_0 \\ |\varphi(u)|_1 \\ \vdots \\ |\varphi(u)|_{d-1} \end{pmatrix} = M_\varphi \begin{pmatrix} |u|_0 \\ |u|_1 \\ \vdots \\ |u|_{d-1} \end{pmatrix}. \tag{1.2}$$

Further, the incidence matrix of a composed morphism satisfies $M_{\varphi\circ\theta} = M_\varphi M_\theta$.

It is important to take into consideration that the incidence matrix does not determine a substitution uniquely.

For a given infinite word, our aim may be to describe its complexity. For this purpose, there are defined various complexity functions which measure different kinds of complexities, for instance factor complexity, Abelian complexity, arithmetical complexity, permutation complexity, palindromic complexity, θ-palindromic complexity, …Now we define the factor complexity whereas the Abelian complexity will be defined in Chapter 3.

Definition 1.6. Let $\boldsymbol{u}$ be an infinite word. The map $\mathcal{C} : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined as

$$\mathcal{C}(n) = \#\mathcal{L}_n(\boldsymbol{u})$$

is called **factor complexity**. Δ

Basic properties of factor complexity are summarized in the following lemma.

Lemma 1.7. The factor complexity of an infinite word $\boldsymbol{u}$ is an increasing function satisfying for all $n \in \mathbb{Z}^+$

- $1 \leq \mathcal{C}(n) \leq (\#\mathcal{A})^n$;
- $\Delta\mathcal{C}(n) = \mathcal{C}(n+1) - \mathcal{C}(n) = \sum_{w\in\mathcal{L}_n(\boldsymbol{u})} (\#Rext(w) - 1)$.

Δ

The most famous class of aperiodic words are Sturmian words, which are defined as aperiodic words with the minimal possible factor complexity, i.e., $\mathcal{C}(n) = n + 1$. It is easily seen that their alphabet is binary. For more information about these words, see [12].

CHAPTER 2

Balances of d-bonacci word

The notion of **balanced words** was already introduced by Morse and Hedlund in [15], where they studied properties of Sturmian words. Its generalization, the so-called c-balanced words, appeared in the work [6]. An infinite word is said to be **c-balanced**, if for any two its factors v and w of the same length, we have $||v|_a - |w|_a| \leq c$ for any letter $a \in \mathcal{A}$. As shown by Adamczewski (in [1, 2]; for a simplified version of the proof, see [4]), any fixed point of a Pisot-type substitution (definition can be found, e.g., in [16]) is c-balanced for some c.

We use techniques of [1, 2, 19] to find an upper estimate on the smallest possible value c for the **d-bonacci word**, i.e., the unique fixed point of the substitution

$$\varphi \quad \begin{cases} \quad 0 & \to \quad 01; \\ \quad 1 & \to \quad 02; \\ & \vdots \\ (d-1) & \to \quad 0(d-1); \\ (d-2) & \to \quad 0; \end{cases} \tag{2.1}$$

with the incidence matrix M_φ. For a given d and the associated d-bonacci word $\boldsymbol{u}$, our aim is to estimate the maximal value of the so-called balance function

$$B_a(n) = \max\left\{|v|_a - |w|_a \mid v, w \text{ are factors of } \boldsymbol{u} \text{ of length } n\right\}, \tag{2.2}$$

where a is from the d-letter alphabet $\mathcal{A} = \{0, 1, \ldots, d-1\}$ and $n \in \mathbb{Z}^+$.

The characteristic polynomial of M_φ is $f(x) = x^d - x^{d-1} - \ldots - 1$. Denote its roots by $\beta = \beta_1,\ \beta_2,\ \ldots,\ \beta_d$ in the way that $|\beta| \geq |\beta_2| \geq \ldots |\beta_d|$. The number β is real, strictly greater than 1 and all other roots are in absolute value strictly less than 1.

The cases which have been already investigated are:

i) $d = 2$ (Fibonacci word) – a representative of the Sturmian words (i.e., aperiodic words having the lowest possible factor complexity) which have constant balance function $B_a(n) = 1$ for both letters a;

ii) $d = 3$ (Tribonacci word) – as proven in [19], $\max_{n\in\mathbb{Z}^+} B_a(n) = 2$ for all letters.

In [19], the conjecture that the balance function of the d-bonacci word is bounded by $d-1$ was stated. We will show that it holds for all $d \in \{2, \ldots, 12\}$ and that this bound can be diminished.

2.1 Upper estimates of $B_a(n)$

Let $\boldsymbol{u}$ be an infinite word. We define the **discrepancy** function in a slightly more general way than it is usual – for an arbitrary factor of the studied word instead of only for its prefixes:

$$D_a(n_1,n_2) = |\boldsymbol{u}_{[n_1,n_2)}|_a - \mu_a(n_2-n_1) = \underbrace{(|\boldsymbol{u}_{[0,n_2)}|_a - \mu_a n_2)}_{=:D_a(n_2)} - \underbrace{(|\boldsymbol{u}_{[0,n_1)}|_a - \mu_a n_1)}_{=:D_a(n_1)}, \tag{2.3}$$

where $\mu_a = \lim_{N\to+\infty} \frac{|\boldsymbol{u}_{[0,N)}|_a}{N}$ is the frequency of the letter a in $\boldsymbol{u}$. For a given factor $\boldsymbol{u}_{[n_1,n_2)}$, the discrepancy function describes how much the actual count of the letter a in the factor does differ from the count expected from the letter frequency μ_a.

Since $B_a(n) = \max\{D_a(n_1,n_2) - D_a(\tilde{n}_1,\tilde{n}_2) \mid n_2 - n_1 = \tilde{n}_2 - \tilde{n}_1\}$, we obtain the following estimate:

$$B_a(n) \leq \left\lfloor 2\underbrace{\left(\sup\left\{D_a(n) \mid n \in \mathbb{Z}^+\right\} - \inf\left\{D_a(n) \mid n \in \mathbb{Z}^+\right\}\right)}_{=:\Delta D_a}\right\rfloor .$$

Now, for all letters $a \in \mathcal{A}$, let us define row vectors

$$f^{(a)} = (0,\ldots,0,\underbrace{1}_{a^{\text{th}}\text{ entry}},0,\ldots 0) - (\mu_a,\ldots,\mu_a)$$

which enable us to express the discrepancy function of prefixes as

$$D_a(n) = f^{(a)}\cdot v,$$

where v is a column vector with components $v_i = |\boldsymbol{u}_{[0,n)}|_i$ for $i \in \{0,\ldots,d-1\}$.

As follows from [10, 18], for all finite integer sequences $(i_j)_{j=0}^N$ such that $i_N > i_{N-1} > \ldots > i_1 > i_0 \geq 0$ and the d-bonacci substitution φ, the word

$$\varphi^{i_N}(0)\varphi^{i_{N-1}}(0)\cdots\varphi^{i_1}(0)\varphi^{i_0}(0) \tag{2.4}$$

is a prefix of the d-bonacci word and any prefix can be written in this form. Hence the discrepancy function of every prefix of the d-bonacci word may be expressed as

$$D_a(n) = \sum_{j=0}^{N} g_{(a,i_j)} = \sum_{j=0}^{i_N} \delta_j g_{(a,j)}$$

for some $(\delta_0,\ldots,\delta_{i_N}) \in \{0,1\}^{i_N+1}$, where

$$g_{(a,j)} = f^{(a)}\cdot M_\varphi^j \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}.$$

Remark that $g_{(a,j)}$ depends on d. The formula provides bounds on $D_a(n)$:

$$\text{sum of negative } g_{(a,i)}\text{'s} \leq D_a(n) \leq \text{sum of positive } g_{(a,i)}\text{'s}.$$

Thus $\Delta D_a \leq \sum_{i=0}^{+\infty} |g_{(a,i)}|$.

Theorem 2.1. The d-bonacci word is c-balanced for

$$c = \max_{a \in \mathcal{A}} \left\lfloor 2 \cdot \sum_{i=0}^{+\infty} |g_{(a,i)}| \right\rfloor .$$

$\triangle$

Among several different ways of expressing $g_{(a,i)}$ (e.g., using Jordan normal form of M_φ), two of them will play an essential role in the following text.

Proposition 2.2. For a given alphabet of cardinality d, a fixed letter $a \in \mathcal{A}$, and $k \geq 0$; it holds:

i)

$$g_{(a,k)} = T_{k+d-a-1} - \frac{T_{k+d}}{\beta^{a+1}}; \tag{2.5}$$

ii)

$$g_{(a,k)} = \sum_{j=2}^{d} \left(\frac{1}{\beta_j^{a+1}} - \frac{1}{\beta^{a+1}} \right) \frac{\beta_j^{k+d}}{f'(\beta_j)}, \tag{2.6}$$

where T_n is defined by the d-bonacci recurrence $T_n = \sum_{i=n-d}^{n-1} T_i$ with the initial conditions $T_0 = T_1 = \ldots = T_{d-2} = 0$ and $T_{d-1} = 1$.

$\triangle$

In order to find an upper bound c, we sum up the first m members of $(|g_{(a,i)}|)_{i=0}^{+\infty}$ and estimate the rest of them:

$$\sum_{i=0}^{+\infty} |g_{(a,i)}| \leq \sum_{i=0}^{m-1} |g_{(a,i)}| + E, \qquad \text{where } E \text{ is arbitrary such that} \qquad E \geq \sum_{i=m}^{+\infty} |g_{(a,i)}|.$$

The second statement of Proposition 2.2 provides setting

$$E_{(a,m)} := |\beta_2|^m \sum_{j=2}^{d} \left| \left(\frac{1}{\beta_j^{a+1}} - \frac{1}{\beta^{a+1}} \right) \frac{1}{f'(\beta_j)} \right| \frac{|\beta_j|^d}{1 - |\beta_j|} = |\beta_2|^m \cdot C(d,a).$$

To conclude, we have to find m great enough to satisfy

$$\left\lfloor 2 \sum_{i=0}^{m-1} \left| g_{(a,i)} \right| \right\rfloor = \left\lfloor 2 \left(\sum_{i=0}^{m-1} \left| g_{(a,i)} \right| + E_{(a,m)} \right) \right\rfloor . \tag{2.7}$$

2.1.1 Computation

We have derived an upper bound for ΔD_a. Since we always compute on machines working in a finite precision, we try to reduce work with non-integer numbers. Therefore, we make use of the fact that, for a fixed letter a and the alphabet cardinality d, the sequence of $g_{(a,i)}$'s satisfies the d-bonacci recurrence (follows, e.g., from (2.5)). Namely, $g_{(a,n+d)} = g_{(a,n+d-1)} + \ldots + g_{(a,n)}$.

	IC of $(g_{(a,i)})_{i=0}^{3}$	$a=0$	$a=1$	$a=2$	$a=3$
$g_{(a,0)}$	$(1,0,0,0)$	$+$	$-$	$-$	$-$
$g_{(a,1)}$	$(0,1,0,0)$	$-$	$+$	$-$	$-$
$g_{(a,2)}$	$(0,0,1,0)$	$-$	$-$	$+$	$-$
$g_{(a,3)}$	$(0,0,0,1)$	$-$	$-$	$-$	$+$
$g_{(a,4)}$	$(1,1,1,1)$	$+$	$-$	$-$	$-$
$g_{(a,5)}$	$(1,2,2,2)$	$-$	$+$	$-$	$-$
$g_{(a,6)}$	$(2,3,4,4)$	$-$	$-$	$+$	$-$
$g_{(a,7)}$	$(4,6,7,8)$	$-$	$-$	$-$	$+$
$g_{(a,8)}$	$(8,12,14,15)$	$+$	$+$	$-$	$-$
$g_{(a,9)}$	$(15,23,27,29)$	$-$	$+$	$+$	$-$
$g_{(a,10)}$	$(29,44,52,56)$	$-$	$-$	$+$	$+$
$g_{(a,11)}$	$(56,85,100,108)$	$+$	$-$	$-$	$+$
$g_{(a,12)}$	$(108,164,193,208)$	$+$	$+$	$-$	$-$

Table 2.1: **4-bonacci** – *$g_{(a,i)}$ as an integer combination of $(g_{(a,0)},\dots,g_{(a,3)})$ and its signum.*

	$a=0$	$a=1$	$a=2$	$a=3$
$\sum_{i=0}^{12}\lvert g_{(a,i)}\rvert$ as IC	$\begin{pmatrix}123\\183\\215\\232\end{pmatrix}$	$\begin{pmatrix}39\\63\\71\\76\end{pmatrix}$	$\begin{pmatrix}-133\\-201\\-233\\-254\end{pmatrix}$	$\begin{pmatrix}-47\\-71\\-83\\-86\end{pmatrix}$
$\sum_{i=0}^{12}\lvert g_{(a,i)}\rvert$ sym.	$1664-\frac{3205}{\beta}$	$286-\frac{1057}{\beta^2}$	$\frac{3499}{\beta^3}-487$	$\frac{1209}{\beta^4}-86$
$\sum_{i=0}^{12}\lvert g_{(a,i)}\rvert$ num.	1.2778	1.5157	1.5611	1.5776
$E_{(a,13)}$	0.20054	0.22213	0.25916	0.31056
$\sum_{i=0}^{12}\lvert g_{(a,i)}\rvert + E$	1.49844	1.76006	1.84618	1.91919
$B_a(n)$ upp. est.	2	3	3	3

Table 2.2: **4-bonacci** – *Estimates of $\sum_{i=0}^{+\infty}\lvert g_{(a,i)}\rvert$ and resulting upper estimates of $B_a(n)$.*

$d \setminus a$	0	1	2	3	4	5	6	7	8	9	10	11
2	1	1	×	×	×	×	×	×	×	×	×	×
3	2	2	2	×	×	×	×	×	×	×	×	×
4	2	3	3	3	×	×	×	×	×	×	×	×
5	2	3	3	3	3	×	×	×	×	×	×	×
6	3	3	4	4	4	4	×	×	×	×	×	×
7	3	4	4	4	4	4	4	×	×	×	×	×
8	3	4	4	4	4	4	4	4	×	×	×	×
9	3	4	5	5	5	5	5	5	5	×	×	×
10	3	5	5	5	5	5	5	5	5	5	×	×
11	4	5	5	6	6	6	6	6	6	6	6	×
12	4	5	6	6	6	6	6	6	6	6	6	6

Table 2.3: *Upper estimates of the balance function $B_a(n)$ of the d-bonacci word for $d \in \{2, \ldots, 11\}$ and the letters $a \in \{0, \ldots, d-1\}$.*

We demonstrate the method on the 4-bonacci word. The first step is calculating[1] $sgn(g_{(a,i)})$ from (2.5) for all $i \in \{0, \ldots, m-1\}$ (illustrated in Table 2.1). Then we express $\sum_{i=0}^{m-1} |g_{(a,i)}|$ as an integer combination (IC) of $(g_{(a,0)}, \ldots, g_{(a,d-1)})$ which can be rewritten into the form $e + \frac{f}{\beta^{a+1}}$ for some $e, f \in \mathbb{Z}$ (as also follows from (2.5)) and then evaluated[1] (illustrated in Table 2.2). The final step is verification of the equality (2.7).

To make our procedure reliable with respect to possible rounding errors, we replace the estimated error $E_{(a,m)}$ by a constant $E > E_{(a,m)}$. If (2.7) holds, it is equal to the desired upper bound of $B_a(n)$ (but it may not be optimal). In the opposite case, we must increase m and repeat the procedure.

Our obtained results for $d \in \{2, \ldots, 12\}$ are summarized in Table 2.3. For detailed tables see Appendix A.

From Theorem 2.1, we have not found a derivation of an upper bound on balance for a general d, yet. It seems from Table 2.3 that it might be, e.g., $\lceil \frac{d+1}{2} \rceil$.

2.2 Lower estimates of $\max_{n \in \mathbb{Z}^+} B_a(n)$

To find lower bounds on the constant c, one needs to find among factors of the d-bonacci word two factors v and w of the same length with $|w|_a - |v|_a$ great enough. Computer searching in the set of all factors is very time-consuming. Therefore, we use similar ideas as in the previous sections.

The equations (2.2) and (2.3) enable us to express balance function as

$$B_a(n) = \max \{ D_a(n_1, n_2) - D_a(\tilde{n}_1, \tilde{n}_2) \mid \underbrace{n = n_2 - n_1 = \tilde{n}_2 - \tilde{n}_1}_{\text{condition } \mathcal{C}} \}.$$

Therefore, we can estimate lower bound of $\max_{n \in \mathbb{Z}^+} B_a(n)$ by

$$|D_a(n_1, n_2) - D_a(\tilde{n}_1, \tilde{n}_2)|$$

[1]The calculation must be performed in an environment working in enough precision, e.g., Wolfram Mathematica.

$d \setminus a$	0	1	2	3	4	5	6	7	8	9	10	11
2	1	1	×	×	×	×	×	×	×	×	×	×
3	2	2	2	×	×	×	×	×	×	×	×	×
4	2	2	2	2	×	×	×	×	×	×	×	×
5	2	3	3	3	3	×	×	×	×	×	×	×
6	2	3	3	3	3	3	×	×	×	×	×	×
7	2	3	3	3	3	3	3	×	×	×	×	×
8	3	3	4	4	4	4	4	4	×	×	×	×
9	3	4	4	4	4	4	4	4	4	×	×	×
10	3	4	4	4	4	4	4	4	4	4	×	×
11	3	4	4	4	4	4	4	4	4	4	4	×
12	3	4	5	5	5	5	5	5	5	5	5	5

Table 2.4: *Lower estimates of maximum of the balance function $B_a(n)$ of the d-bonacci word for $d \in \{2, \dots, 11\}$ and $a \in \{0, \dots, d-1\}$.*

with arbitrary non-negative n_1, n_2, $\tilde{n}_1$, and $\tilde{n}_2$ satisfying the condition $\mathcal{C}$. Our method how to set them is based on finding a factor being very "deviated" in count of the letter a – we will denote such factor by w.

Since positive $g_{(a,i)}$'s correspond to prefixes containing many a's and negative $g_{(a,i)}$'s correspond to those with few a's, we can obtain n_1 and n_2 in the following way:

$$\begin{aligned}
\bar{n}_1 &:= \sum_{i=0}^{i_{max}} \begin{cases} |\varphi^i(0)| & \text{if } g_{(a,i)} \text{ is positive} \\ 0 & \text{if } g_{(a,i)} \text{ is negative} \end{cases} \\
\bar{n}_2 &:= \sum_{i=0}^{i_{max}} \begin{cases} 0 & \text{if } g_{(a,i)} \text{ is positive} \\ |\varphi^i(0)| & \text{if } g_{(a,i)} \text{ is negative} \end{cases} \\
n_1 &:= \min\{\bar{n}_1, \bar{n}_2\} \\
n_2 &:= \max\{\bar{n}_1, \bar{n}_2\}
\end{aligned}$$

Thus we obtain $w = \boldsymbol{u}_{[n_1,n_2)}$. Note that there could appeare two situations:

$$\boldsymbol{u} = \overbrace{\underbrace{\boldsymbol{u}_0 \cdots \boldsymbol{u}_{n_1-1}}_{\text{prefix with many } a\text{'s}} \quad \underbrace{\boldsymbol{u}_{n_1} \cdots \boldsymbol{u}_{n_2-1}}_{w \text{ (i.e., factor with few } a\text{'s)}}}^{\text{prefix with few } a\text{'s}} \boldsymbol{u}_{n_2} \cdots\cdots,$$

$$\boldsymbol{u} = \overbrace{\underbrace{\boldsymbol{u}_0 \cdots \boldsymbol{u}_{n_1-1}}_{\text{prefix with few } a\text{'s}} \quad \underbrace{\boldsymbol{u}_{n_1} \cdots \boldsymbol{u}_{n_2-1}}_{w \text{ (i.e., factor with many } a\text{'s)}}}^{\text{prefix with many } a\text{'s}} \boldsymbol{u}_{n_2} \cdots\cdots.$$

Then we choose the right neighbor of w of the same length to be the other factor (in fact, we have considered two more factors, the left neighbor of w in $\boldsymbol{u}$ and the prefix of $\boldsymbol{u}$, but they provide worse results).

Results obtained by this procedure for $d \in \{2, \dots, 12\}$ are summarized in Table 2.4. For detailed tables with the positions of the factors in $\boldsymbol{u}$, see Appendix A.

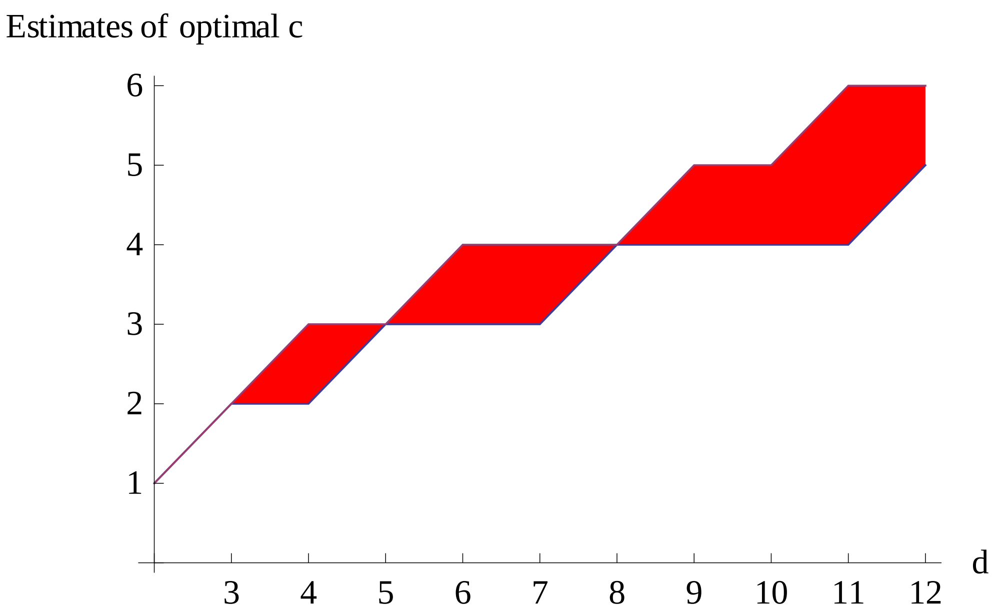

Figure 2.1: *Illustration of lower and upper estimates of* $\max_{a\in\mathcal{A},n\in\mathbb{Z}^+} B_a(n)$ *for the* d*-bonacci word for* $d \in \{2,\ldots,12\}$*. For every* d*, the optimal* c *must be situated in the red area.*

2.3 Summary

As follows from Table 2.3 and Table 2.4, we have obtained the optimal c-balances for the 5-bonacci and for the 8-bonacci.

Theorem 2.3.

- The 5-bonacci word is c-balanced with $c = 3$ and this bound cannot be improved.
- The 8-bonacci word is c-balanced with $c = 4$ and this bound cannot be improved.

$\triangle$

Moreover, for any given $d \geq 4$, we know a pair of factors v and w such that $|v|_a - |w|_a = 3$ for all $a \in \{2,\ldots,d-1\}$. Therefore, we can conclude with the following theorem.

Theorem 2.4 ([5])**.** The 4-bonacci word is c-balanced with $c = 3$ and this bound cannot be improved. $\triangle$

CHAPTER 3

Abelian complexity

The idea of Abelian complexity appeared first in the work [8] by Coven and Hedlund, where the periodic words and the Sturmian words were characterized using the so-called **Parikh vectors**, i.e., vectors counting individual letters in a finite word v:

$$\Psi(v) = (|v|_0, \dots, |v|_{d-1}) \,.$$

However, the first "complete" definition of Abelian complexity as an Abelian version of factor complexity was introduced by Richomme, Saari, and Zamboni in [20]. Then there appeared several works (namely [3, 4, 7, 9, 13, 19, 22, 23, 24, 25]) studying Abelian complexity from various points of view.

Definition 3.1. Let $\boldsymbol{u}$ be an infinite word. Then **Abelian complexity** is the map $\mathcal{AC} : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined as

$$\mathcal{AC}(n) = \#\{\Psi(w) \mid w \in \mathcal{L}_n(\boldsymbol{u})\}.$$

Further, we say that two finite words v and w are **Abelian equivalent** if $\Psi(v) = \Psi(w)$, and we denote it by $v \sim_{ab} w$. △

Now Abelian complexity can be rewritten as

$$\mathcal{AC}(n) = \#\left(\mathcal{L}_n / \sim_{ab}\right).$$

In [20], there were also presented basic properties of $\mathcal{AC}(n)$. Let us mention some of them. If we work with the alphabet $\mathcal{A} = \{0, \dots, d-1\}$, then $1 \leq \mathcal{AC}(n) \leq \binom{n+d-1}{d-1}$. An infinite word $\boldsymbol{u}$ is periodic of period p if and only if $\mathcal{AC}(p) = 1$. For binary words, we have $\mathcal{AC}(n) = B_0(n) + 1 = B_1(n) + 1$. Hence an aperiodic infinite word $\boldsymbol{u}$ is Sturmian if and only if $\mathcal{AC}(n) = 2$ for all positive integers n. Further, Abelian complexity of a word $\boldsymbol{u}$ is bounded if and only if $\boldsymbol{u}$ is c-balanced for some c.

Concerning existence of a bound of $\mathcal{AC}(n)$, we get the following criterium as follows from Chapter 2 (more details can be found in [4]).

Theorem 3.2 ([1, 2, 20])**.** Fixed points of Pisot-type substitutions have bounded Abelian complexity. △

Widmer provided an upper estimate on Abelian complexity from knowledge of c in the c-balance property.

d	n_{max}	$\{\mathcal{AC}(n) \mid n \leq n_{max}\}$	$\left(\max_{n\leq n_{max}} B_0(n), \ldots, \max_{n\leq n_{max}} B_{d-1}(n)\right)$
3	15000	$\{3, \ldots, 7\}$	$(2, 2, 2)$
4	49999	$\{4; 6, \ldots, 15\}$	$(2, 3, 3, 3)$
5	15000	$\{5; 8; 10, \ldots, 31\}$	$(2, 3, 3, 3, 3)$
6	15000	$\{6; 10; 13; 15, \ldots, 18; 20, \ldots, 61\}$	$(2, 2, 3, 3, 3, 2)$
7	15000	$\{7; 12; 16; 19; 21, \ldots, 23; 27; 28; 31 \ldots 115; 118; 119\}$	$(2, 2, 3, 3, 2, 2, 2)$
8	15000	$\{8; 14; 19; 23; 26; 28, \ldots, 30; 35; 39; 40; 43, \ldots, 47; 49, \ldots, 186; 188, \ldots, 194; 196; 197\}$	$(2, 2, 3, 2, 2, 2, 2, 2)$
9	5000	$\{9; 16; 22; 27; 31; 33; 34; 36, \ldots, 38; 42; 44; 47; 49; 54; 57, \ldots, 60; 63, , 67; 69, , 75; 78, , 80; 82, , 241; 243, \ldots, 245; 247, , 250; 252; 254; 257; 258; 264; 266\}$	$(2, 2, 2, 2, 2, 2, 2, 2, 2)$
10	5000	$\{10; 18; 25; 31; 36; 38; 40; 43; 45, \ldots, 47; 49; 55; 58; 64; 65; 69; 70; 73, \ldots, 76; 80; 85; 88; 90; 91; 93, \ldots, 97; 100; 103; 105; 106; 109, \ldots, 133; 136, \ldots, 142; 144, \ldots, 146; 148, \ldots, 161; 163, \ldots, 265; 267, \ldots, 292; 294, \ldots, 297; 299, \ldots, 305; 307, \ldots, 344; 346, \ldots, 356; 358, \ldots, 366; 373; 374; 383; 385, \ldots, 388; 390; 395; 398\}$	$(2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$

Table 3.1: *Values of the Abelian complexity $\mathcal{AC}(n)$ and the balance function $B_a(n)$ of the d-bonacci word for $d \in \{3, \ldots, 10\}$, the factor lengths $n \in \{1, \ldots, n_{max}\}$, and the letters $a \in \{0, \ldots, d-1\}$.*

Theorem 3.3 ([25])**.** Let $\boldsymbol{u}$ be a c-balanced word over the alphabet $\{0, \ldots, d-1\}$. Then for all $n > 0$, it is satisfied

$$\mathcal{AC}(n) \leq \binom{d}{\lfloor \frac{cd}{2} \rfloor}^{[c+1]},$$

where $\binom{k}{m}^{[l]}$ is defined recursively:

$$\binom{k}{m}^{[l+1]} = \sum_{i=0}^{m} \binom{k}{m-i} \binom{m-i}{i}^{[l]} \qquad \text{for } l > 1$$

with the initial condition

$$\binom{k}{m}^{[2]} = \binom{k}{m}.$$

Δ

3.1 d-bonacci word

Let us consider the d-bonacci word, i.e., the unique fixed point of the substitution (2.1).

d	$\mathcal{AC}(n)$ upp. est.
2	2
3	7
4	44
5	155
6	1751
7	8135
8	38165
9	767394
10	4395456
11	117224317
12	786588243

Table 3.2: *Upper estimates of the Abelian complexity $\mathcal{AC}(n)$ of the d-bonacci word for $d \in \{2, \ldots, 11\}$.*

For $d = 2$, we obtain the famous Fibonacci word which is Sturmian (thus balanced) and must have the Abelian complexity constantly equal to 2.

The case $d = 3$, i.e., the Tribonacci word, was studied in articles [19, 24]. In [19], Richomme, Saari, and Zamboni showed that $\mathcal{AC}(n) \in \{3, \ldots, 7\}$ for all n. Moreover, they proved that the values 3 and 7 are attained for infinitely many n's and described all n's for which $\mathcal{AC}(n) = 3$. In [24], Turek presented a method of calculating $\mathcal{AC}(n)$ using the so-called relative Parikh vectors. Using it on the Tribonacci word, he proved that every value from the set $\{4, 5, 6\}$ is attained for infinitely many n's, too.

The Abelian complexity of the d-bonacci word for $d \geq 4$ still remains almost unknown. In Table 3.1, we present results obtained by our computer program. We examined all factors up to some length n_{max}. From the results, one may conjecture that for these d's, there exist some gaps in the range of $\mathcal{AC}(n)$.

Further, in Table 3.2, there are shown upper estimates of $\mathcal{AC}(n)$ derived by Theorem 3.3 from our numerically obtained estimates on balances, as they are presented in Chapter 2. Remark that the values are overestimated.

3.2 Thue-Morse word

In [20], there was studied the so-called Thue-Morse word $\boldsymbol{TM}_{(0)}$, i.e., the fixed of the substitution

$$\varphi \quad \begin{cases} 0 & \rightarrow & 01, \\ 1 & \rightarrow & 10 \end{cases}$$

which is beginning from the letter 0. The Abelian complexity of the word safisfies

$$\begin{aligned} AC(n) &= 2 \qquad \text{for odd } n\text{'s}, \\ AC(n) &= 3 \qquad \text{for even } n\text{'s}. \end{aligned}$$

Moreover, the authors fully characterized all words, which share the same Abelian complexity and the same factor complexity with the Thue-Morse word.

3.3 Words associated with quadratic Parry numbers

In [3], we presented a formula for computing the Abelian complexity of the words associated with quadratic non-simple and simple Parry numbers, i.e., the unique fixed points of the substitutions

$$\varphi_1 \quad \begin{cases} 0 & \to & 0^p1, \\ 1 & \to & 0^q1, \end{cases} \quad \text{where } p > q > 0;$$

and

$$\varphi_2 \quad \begin{cases} 0 & \to & 0^p1, \\ 1 & \to & 0^q, \end{cases} \quad \text{where } p \geq q > 0;$$

respectively.

3.4 Some other words

In [23], Turek considered a word over a ternary alphabet which is the unique fixed point of the substitution

$$\varphi \quad \begin{cases} 0 & \to & 0^p1, \\ 1 & \to & 2, \\ 2 & \to & 0^{p-1}1, \end{cases} \quad \text{where } p > 1.$$

These words correspond to some of cubic Pisot numbers. He proved that such words are 3-balanced. More specifically,

$$\begin{aligned} \max_{n\in\mathbb{Z}^+} B_0(n) &= 3, \\ \max_{n\in\mathbb{Z}^+} B_1(n) &= 2, \\ \max_{n\in\mathbb{Z}^+} B_2(n) &= 2. \end{aligned}$$

Then he showed that $\mathcal{AC}(n) \in \{3, \ldots, 7\}$ for all n. Moreover, for a fixed p, $\mathcal{AC}(n) = 7$ holds for infinitely many n.

Infinite words having unbounded Abelian complexity have not been studied, yet. The only exception is the paper [13], where a paperfolding word is considered and the authors proved that the sequence $(\mathcal{AC}(n))_{n=1}^{+\infty}$ corresponding to such word is 2-regular.

CHAPTER 4

Abelian returns and semi-Abelian returns

The notion of semi-Abelian returns and Abelian returns was introduced by Puzynina and Zamboni in [17] as an Abelian version of the "standard" return words (as defined in Chapter 1). The authors fully characterized the Sturmian words (i.e., balanced aperiodic infinite words) using this notion. Further, in [21], Rigo, Salimov, and Vandomme studied semi-Abelian returns in some infinite words to their proper prefixes. Remark that, despite using semi-Abelian returns, the last authors call it Abelian returns. Nevertheless, we will keep the notion of Puzynina and Zamboni.

Let us recall the basic definitions.

Definition 4.1 ([17])**.** Let $\boldsymbol{u}$ be an infinite word and let v be its finite factor of length n. Consider all $n_1 < n_2 < \dots$ such that $v \sim^{ab} \boldsymbol{u}_{[n_i, n_i+n)}$. For all possible i:

- the factor $\boldsymbol{u}_{[n_i, n_{i+1})}$ is said to be **semi-Abelian return** of the factor v in $\boldsymbol{u}$;
- the Parikh vector $\Psi\left(\boldsymbol{u}_{[n_i, n_{i+1})}\right)$ is said to be **Abelian return** of the factor v in $\boldsymbol{u}$.

Δ

It is seen from the definition that semi-Abelian returns and Abelian returns in a given infinite word depend only on the Parikh vector of v.

Definition 4.2 ([21])**.** For an infinite word $\boldsymbol{u}$, we denote the set of all semi-Abelian returns to all its proper finite prefixes by $SAPR(\boldsymbol{u})$. Δ

4.1 Sturmian words

The Sturmian words can be fully characterized by the number of their (semi-)Abelian returns.

Theorem 4.3 ([17])**.** i) A binary recurrent infinite word $\boldsymbol{u}$ is Sturmian if and only if each factor v of $\boldsymbol{u}$ has two or three semi-Abelian returns in $\boldsymbol{u}$.

ii) A binary recurrent infinite word $\boldsymbol{u}$ is Sturmian if and only if each factor v of $\boldsymbol{u}$ has two or three Abelian returns in $\boldsymbol{u}$.

$\triangle$

In this text, we provide a proof of the implications from the left to the right (i.e., every finite factor of a Sturmian word has two or three (semi-)Abelian returns in the Sturmian word) rewritten in detail (since in [17], the original proof was partially abbreviated). As an auxiliary tool, a theory connected with balanced binary periodic words is used. Remark that a way of proving is the same as in [17].

4.1.1 Balanced finite binary words

To extend the definition of balanced words from Chapter 2, we say that a finite word v is balanced if v^ω is balanced. Such words were characterized in [11]. Let us recall the most important definitions and results.

We will use **shift** operation $\sigma : \mathcal{A}^n \to \mathcal{A}^n$ defined as $\sigma(v_0 v_1 \ldots v_{n-1}) = v_1 \ldots v_{n-1} v_0$. Words v and w such that $v = \sigma^i(w)$ for some integer i are said to be **cyclically conjugated** and we write $v \sim_\sigma w$. It is readily seen that the relation $\sim_\sigma$ is an equivalence.

For given co-primes $p < q$, we define the following sets:

$$\begin{aligned} \mathcal{W}_{p,q} &= \{v \in \{0,1\}^q \mid |v|_1 = p\}\,, \\ \mathbb{W}_{p,q} &= \mathcal{W}_{p,q} / \sim_\sigma . \end{aligned}$$

Elements of $\mathbb{W}_{p,q}$ are called **orbits**.

Assume $|w| = q$ and $|w|_1 = p$, where p and q are co-primes. We denote the unique orbit containing w by $[w]$. It follows from $gcd(p,q) = 1$ that $[w]$ contains q words. After ordering all elements of $[w]$ lexicographically, let us denote them in the following way:

$$w^{(0)} <_L w^{(1)} <_L \ldots <_L w^{(q-1)}.$$

Then we define the so-called **lexicographic array** $A[w] \in \{0,1\}^{q,q}$ as

$$A[w] = \begin{pmatrix} w^{(0)} \\ w^{(1)} \\ \vdots \\ w^{(q-1)} \end{pmatrix},$$

i.e., a matrix having the words $w^{(0)}, \ldots, w^{(q-1)}$ as the rows.

In order to make the following text more comprehensible, indexes of rows and columns of $A[w]$ and indexes i and j in $\left(w^{(i)}\right)_j$ are considered to be arbitrary integers (not only from the set $\{0, \ldots, q-1\}$) and we take them modulo q, i.e., $A[w]_{i,j} = \left(w^{(i)}\right)_j = \left(w^{(i \mod q)}\right)_{j \mod q}$. Further, instead of writing $\left(w^{(i)}\right)_j$, we will use shorter $w_j^{(i)}$.

The following theorem provides a procedure how to decide whether a given finite binary word w is balanced or not.

Theorem 4.4 ([11])**.** For a given $w \in \{0,1\}^q$, the following statements are equivalent:

i) w is balanced;

ii) $\left|w^{(i)}_{[0,j)}\right|_1 \leq \left|w^{(i+1)}_{[0,j)}\right|_1$ for all $i \in \{0,\ldots,q-2\}$ and $j \in \{1,\ldots,q\}$.

$\triangle$

Let us mention an example of a balanced and of an unbalanced word.

Example 4.5. $w = 0101010$:

$$\overbrace{0010101}^{w^{(0)}} <_L \overbrace{0100101}^{w^{(1)}} <_L \overbrace{0101001}^{w^{(2)}} <_L \overbrace{0101010}^{w^{(3)}} <_L \overbrace{1001010}^{w^{(4)}} <_L \overbrace{1010010}^{w^{(5)}} <_L \overbrace{1010100}^{w^{(6)}}$$

$$A[w] = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We see that w is balanced.

$\triangle$

Example 4.6. $w = 0101100$:

$$\overbrace{0001011}^{w^{(0)}} <_L \overbrace{0010110}^{w^{(1)}} <_L \overbrace{0101100}^{w^{(2)}} <_L \overbrace{0110001}^{w^{(3)}} <_L \overbrace{1000101}^{w^{(4)}} <_L \overbrace{1011000}^{w^{(5)}} <_L \overbrace{1100010}^{w^{(6)}}$$

$$A[w] = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$|w^{(3)}_{[0,3)}|_1 > |w^{(4)}_{[0,3)}|_1$, *hence w is balanced.* $\triangle$

The following two theorems provide us an easy way how to construct a lexicographic array on one hand (a construction using shifts of the word $0^{|v|_0}1^{|v|_1}$) and a way how to verify the balance property of a finite word v with $gcd(|v|, |v|_1) = 1$ on the other hand (we construct the associated lexicographic array and check if v is a row of the array).

Theorem 4.7 ([11])**.** The orbit set $\mathbb{W}_{p,q}$ contains one and only one balanced orbit. $\triangle$

Theorem 4.8 ([11])**.** Let $[w]$ be the unique balanced orbit in $\mathbb{W}_{p,q}$. Define $\tilde{v} = 0^{q-p}1^p$. Then for all integers $i, j \in \{0,\ldots,q-1\}$, we have:

i) $A[w]_{\bullet,j} = (\sigma^{jp}(\tilde{v}))^{\intercal}$, where $A[w]_{\bullet,j}$ is the column number j of the matrix $A[w]$;

ii) $A[w]_{i,\bullet} = w^{(i)} = \tilde{v}_i(\sigma^p(\tilde{v}))_i(\sigma^{2p}(\tilde{v}))_i \cdots (\sigma^{(q-1)p}(\tilde{v}))_i$, where $A[w]_{i,\bullet}$ is the row number i of the matrix $A[w]$.

$\triangle$

4.1.2 Usage on Sturmian words

We say that a finite factor v of a Sturmian word $\boldsymbol{u}$ is a **standard factor** if v is a letter or $v = Bab$, where $a \neq b \in \mathcal{A}$ and B is a bispecial factor of $\boldsymbol{u}$.

First we show how to find all semi-Abelian returns in a Sturmian word $\boldsymbol{u}$. Let v be an arbitrary finite factor of $\boldsymbol{u}$ of length n. Find a standard factor w of $\boldsymbol{u}$ long enough to contain v and all its semi-Abelian returns. As proven in [14] in terms of the Burrows-Wheeler transformation, $\sigma^i(w)$ is a factor of $\boldsymbol{u}$ for all i and $gcd(p,q) = 1$, where $|w| = q$, $|w|_1 = p$. Moreover, it implies that w is balanced and v has the same semi-Abelian returns in $\boldsymbol{u}$ as in w^ω.

If we take the matrix $A[w]_{\bullet,0\ldots(n-1)}$ (i.e., matrix consisting of the first n columns of the lexicographic array $A[w]$), its rows are members of two Abelian classes as follows from the balance property of w. Without loss of generality, assume that v is in the Abelian class containing more 0's (otherwise switch 0's and 1's).

With respect to the second statement of Theorem 4.4 and the balance property of w, there exists $m \in \{0,\ldots,q-2\}$ such that for all $i \in \{0,\ldots,m-1\}$, the word $w^{(i)}_{[0,n)}$ is in the same Abelian class as v, i.e., $w^{(i)} \sim_{ab} v$ and for all $i \in \{m,\ldots,q-1\}$ not, i.e., $w^{(i)} \not\sim_{ab} v$.

For all $i \in \{0,\ldots,m-1\}$, find the smallest $r_i > 0$ such that

$$w^{(i)}_{[r_i,r_i+n)} \sim_{ab} v.$$

Then

$$\left\{ w^{(i)}_{[0,r_i)} \mid i \in \{0,\ldots,m-1\} \right\}$$

is the set of all semi-Abelian returns of v in $\boldsymbol{u}$.

Example 4.9 (Fibonacci word $\boldsymbol{f} = 0100101001001010010100100101 0\cdots$, factor $v = 0100$)**.** *A desired standard factor containing v and all its semi-Abelian returns is $w = 0100101001001$. We write down its lexicographic array and, by the procedure above, we mark all the semi-Abelian returns.*

$$A[w] = \left(\begin{array}{cccc|ccccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline\hline
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}\right)$$

For all $i \in \{0, \dots, m-1\}$, the words $w^{(i)}_{[r_i, r_i+n)}$ are in red and the obtained semi-Abelian returns in blue. Hence the semi-Abelian returns of 0010 *in the Fibonacci word are* 0, 01, *and* 001. △

The following lemmas will help us in the proof of the main theorem. The first one describes a structure of all semi-Abelian returns in the Sturmian words. The second one shows that lexicographic arrays are invariant under some indexes changes in this case.

Lemma 4.10. Semi-Abelian returns of a Sturmian word $\boldsymbol{u}$ are letters or they take the form aBb, where a and b are letters, $a \neq b$, and B is a bispecial factor of $\boldsymbol{u}$. △

Proof. Let v be an arbitrary finite factor of $\boldsymbol{u}$, where $\boldsymbol{u}_{[i,i+n)} = v$. Then the following cases should be considered:

i) $\boldsymbol{u}_i = \boldsymbol{u}_{i+n}$:
Since $\boldsymbol{u}_{[i,i+n)} \sim_{ab} \boldsymbol{u}_{[i+1,i+n+1)}$, the letter $\boldsymbol{u}_i$ is a semi-Abelian return of v.

ii) $\boldsymbol{u}_i \neq \boldsymbol{u}_{i+n}$:
Denote the letters $\boldsymbol{u}_i$ and $\boldsymbol{u}_{i+n}$ by a and b respectively. Find $\ell > 0$ smallest possible such that $u_{i+\ell} \neq u_{i+n+\ell}$ and denote $x = \boldsymbol{u}_{[i+1,i+\ell)}$. Then $\boldsymbol{u}_{i+\ell} = b$ and $\boldsymbol{u}_{i+n+\ell} = a$, otherwise $|\boldsymbol{u}_{[i,i+\ell+1)}|_a - |\boldsymbol{u}_{[i+n,i+n+\ell+1)}|_a = 2$, which would be a contradiction with the balance property of $\boldsymbol{u}$. The factor axb is a semi-Abelian return of the factor v in $\boldsymbol{u}$ because $v \sim_{ab} \boldsymbol{u}_{[i+\ell+1,i+n+\ell+1)}$ and ℓ is the smallest possible. Moreover, x is a bispecial factor of $\boldsymbol{u}$.

$\cdots$	v				$\cdots$			
$\cdots$	a	x	b	$\cdots$	b	x	a	$\cdots$
$\cdots$	$\boldsymbol{u}_i$	$\cdots$	$\boldsymbol{u}_{i+\ell}$	$\cdots$	$\boldsymbol{u}_{i+n}$	$\cdots$	$\boldsymbol{u}_{i+n+\ell}$	$\cdots$

□

Corollary 4.11. Let $\boldsymbol{u}$ be a Sturmian word and v its finite factor. Then v has at most one semi-Abelian return of length $l > 1$. △

Proof. Consider words y and z to be semi-Abelian returns of v, where $|y| = |z| > 1$. Then $y = aB_1b$ and $z = aB_2b$, where a and b are letters. Since in a Sturmian word, there exists one and only one bispecial factor of every length, thus $B_1 = B_2$ must hold. □

Lemma 4.12. Let $A[w]$ be a lexicographic array from Theorem 4.8. Consider integers i_0 and j_0 such that $w^{(i_0-1)}_{j_0} = 1$ and $w^{(i_0)}_{j_0} = 0$. Then

$$A[w]_{i,j} = A[w]_{i+i_0,j+j_0} \qquad \text{(i.e, } w^{(i)}_j = w^{(i+i_0)}_{j+j_0}\text{)}$$

for all integers i and j. △

Proof. It is a simple corollary of Theorem 4.8. □

Example 4.13.

$$A[w] = \left(\begin{array}{c|cccccc} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

$(i_0, j_0) = (4, 1)$, *other possible choices of* (i_0, j_0) *could be* $(0, 0)$, $(1, 2)$, $(2, 4)$, $(3, 6)$, $(5, 3)$, *or* $(6, 5)$. Δ

The following observation provides positions of all factors from the Abelian class of v in a lexicographic array.

Observation 4.14. Let $A[w]$, m, v, and n be from the procedure of the semi-Abelian returns searching. Let $w_{j_0}^{(i_0-1)} = 1$ and $w_{j_0}^{(i_0)} = 0$. Then for all $i \in \{i_0, \ldots, i_0 + m - 1\}$, the word $w_{[j_0, j_0+n)}^{(i)}$ is in the same Abelian class as v, i.e.,

$$w_{[j_0, j_0+n)}^{(i)} \sim_{ab} v.$$

Therefore, in every column of $A[w]$, there start exactly m words from the Abelian class of v and they are in consecutive rows. Δ

Example 4.15 (Fibonacci word, $v = 0100$)**.** *We find* $n = 4$ *and* $m = 6$. *For the option* $(i_0, j_0) = (6, 4)$, v *is in blue and all factors starting in the column* 4 *and being from the Abelian class of* v *are in red.*

$$A[w] = \left(\begin{array}{cccc|ccccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline\hline
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}\right)$$

Δ

Theorem 4.16 ([17], a weaker version of Theorem 4.3)**.** Let $\boldsymbol{u}$ be a Sturmian word and v its finite factor. Then:

i) v has two or three semi-Abelian returns in $\boldsymbol{u}$;

ii) v has two or three Abelian returns in $\boldsymbol{u}$.

Δ

Proof. First remark that $\boldsymbol{u}$ cannot have only two semi-Abelian returns for all factors. For a contradiction suppose that there exists a Sturmian word $\boldsymbol{u}$ containing only factors having two semi-Abelian returns. Then we could choose a factor x containing both the letters 0 and 1. Let $x = \boldsymbol{u}_{[p,p+|v|)} = \boldsymbol{u}_{[q,q+|v|)}$ for some $p < q$. It is possible to find some because $\boldsymbol{u}$ is recurrent. The factor $\boldsymbol{u}_{[p,q)}$ has, by the assumption, two semi-Abelian returns so it must

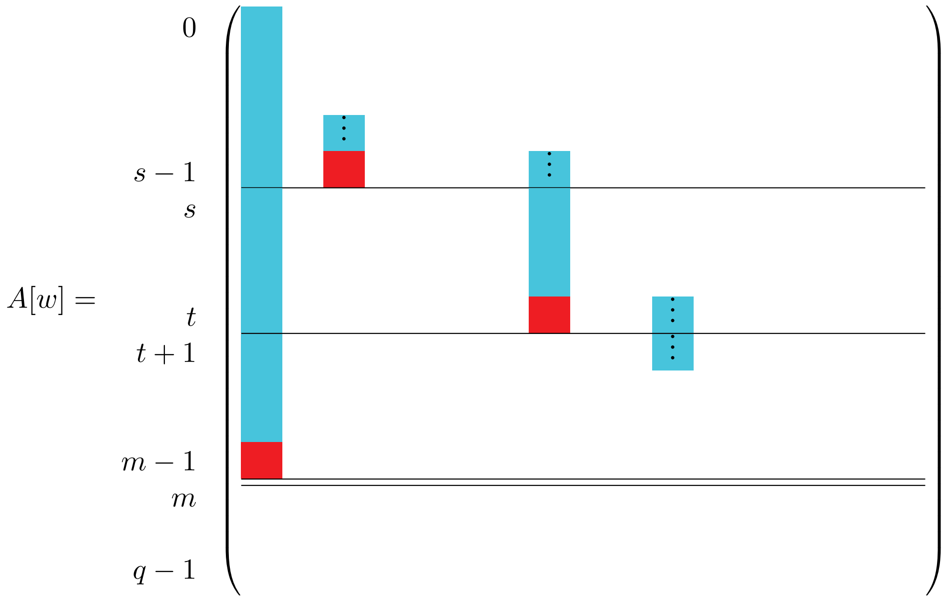

Figure 4.1: *A schematic illustration of the lexicographic array $A[w]$ in the case 1I from the proof of Theorem 4.16 (the highlighted elements correspond to the starting points of row vectors from the Abelian class of v; the elements highlighted in red serve as the "synchronization points").*

be the letters 0 and 1. It is contradicting the fact that $\boldsymbol{u}$ is aperiodic because $\boldsymbol{u}_p\boldsymbol{u}_{p+1}\boldsymbol{u}_{p+2}\cdots$ should be periodic of period $q-p$ since $\mathcal{AC}(q-p)=1$.

Now we will prove that every factor of $\boldsymbol{u}$ has at most 4 semi-Abelian returns.

Take a lexicographic array $A[w]$ by the procedure above. Remark once again that all its indexes are taken modulo q and that $r_i > 0$ are the smallest integers satisfying $w^{(i)}_{[r_i,r_i+n)} \sim_{ab} v$ for all $i \in \{0,\ldots,m-1\}$.

We have to consider the following two cases.

1) *0 or 1 is not a semi-Abelian return of v.*

Suppose, to the contradiction, that v has more than three semi-Abelian returns. With respect to Corollary 4.11, all the semi-Abelian returns must have different lengths.

There cannot exist integers s and t with $0 < s \le t < m-1$ satisfying

$$r_{s-1} > r_s = r_{s+1} = \ldots = r_t < r_{t+1}.$$

Otherwise, by Observation 4.14, r_{s-1} or r_{t+1} would not be the smallest possible and it would be a contradiction.

Together with the fact that there are more than three semi-Abelian returns, it implies that there must exist integers s and t, such that $0 < s \le t < m-1$, satisfying one of the following conditions.

I) $r_{s-1} < r_s = r_{s+1} = \ldots = r_t < r_{t+1}$ (the situation is illustrated in Figure 4.1):

With respect to Observation 4.14, we can use $(m-1,0)$, $(s-1,r_{s-1})$, and (t,r_t) as "synchronization points" in $A[w]$ for gaining shift between the columns 0, r_{s-1},

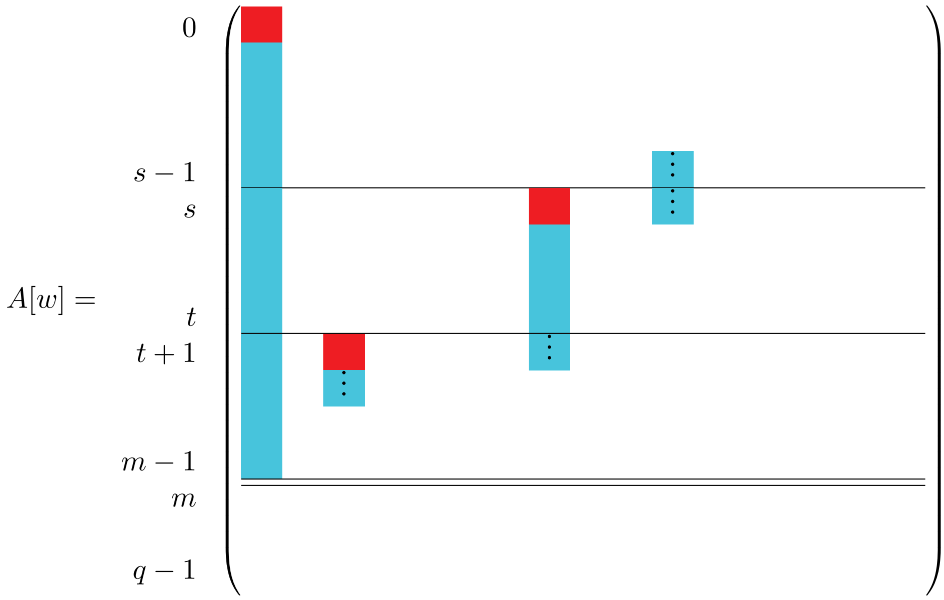

Figure 4.2: *A schematic illustration of the lexicographic array $A[w]$ in the case 1II from the proof of Theorem 4.16 (the highlighted elements correspond to the starting points of row vectors from the Abelian class of v; the elements highlighted in red serve as the "synchronization points").*

and r_t. Then we set

$$\begin{array}{rclcrcl} i_0 & := & s-m, & & j_0 & := & r_{s-1}, \\ k_0 & := & t-m+1, & & l_0 & := & r_t. \end{array}$$

Since

$$\begin{array}{rclcl} A[w]_{i_0-1,j_0} & = & A[w]_{k_0-1,l_0} & = & 1, \\ A[w]_{i_0,j_0} & = & A[w]_{k_0,l_0} & = & 0, \end{array}$$

we can use Lemma 4.12 on (i_0, j_0) and (k_0, l_0) to obtain

$$w^{(s-1)}_{[r_{s-1},r_{s-1}+q)} = w^{(t)}_{[r_t,r_t+q)}.$$

Hence the following equation holds:

$$w^{(s-1)}_{[0,r_{s-1})} = w^{(t)}_{[r_t-r_{s-1},r_t-r_{s-1}+n)}.$$

The same words must be also in the same Abelian class, therefore, we have found $\tilde{r}_t := r_t - r_{s-1} > 0$, which is contradicting the fact that r_t is the smallest possible.

II) $r_{s-1} > r_s = r_{s+1} = \ldots = r_t > r_{t+1}$ (the situation is illustrated in Figure 4.2):
We will use the same way as by the previous point. The only differences are:

- the "synchronization points": $(0,0)$, (s, r_s), and $(t+1, r_{t+1})$
- (i_0, j_0) and (k_0, l_0) for Lemma 4.12:

$$\begin{array}{rclcrcl} i_0 & := & s, & & j_0 & := & r_s, \\ k_0 & := & t+1, & & l_0 & := & r_{t+1}, \end{array}$$

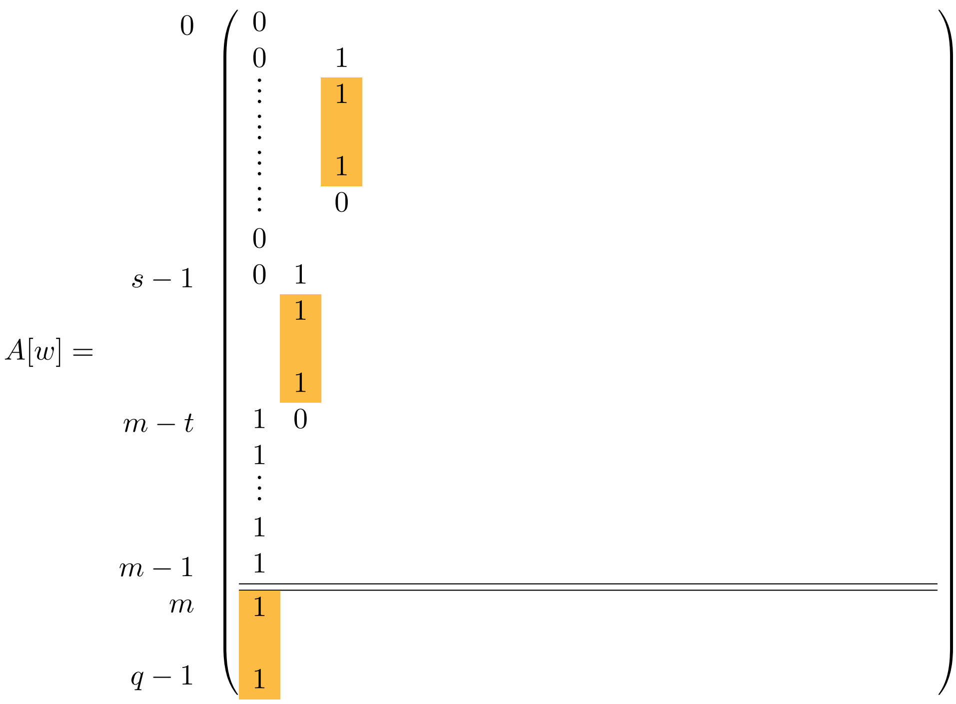

Figure 4.3: *A schematic illustration of the lexicographic array $A[w]$ in the case 2 from the proof of Theorem 4.16 (the highlighted elements correspond to the starting points of row vectors from the unique non-v Abelian class).*

- and the equation

$$w^{(s)}_{[r_s - r_{t+1}, r_s - r_{t+1} + n)} = w^{(t+1)}_{[r_{t+1}, r_{t+1}+n)}$$

  obtained from the lemma.

The equation contradicts the fact that r_s is the smallest one.

All semi-Abelian returns have pairwise distinct lengths, thus the number of semi-Abelian returns must be same as the number of Abelian returns.

2) *Both* 0 *and* 1 *are semi-Abelian returns of* v (the situation is illustrated in Figure 4.3).

Find integers $s, t \in \{1, \ldots, m-1\}$ such that the rows $0, \ldots, s-1$ and $m-t, \ldots, m-1$ correspond to the semi-Abelian return words 0 and 1 respectively. Number of rows corresponding to the unique non-v Abelian class must be $q - m$.

In the column 0, factors of the non-v Abelian class start in the rows $\{m, \ldots, q-1\} =: M_0$. Similarly, in the column 1, factors of the non-v Abelian class start in the rows $\{s, \ldots, m-t-1\} =: M_1$. In the same way, we denote other such rows' number sets by M_i. It is satisfied $M_i \bigcap M_{i+1} = \emptyset$ for all possible i. Hence also $M_1 \bigcap M_2 = \emptyset$. Therefore, length of other longer semi-Abelian returns must be 2. Since, by Lemma 4.10, such semi-Abelian returns can be only 01 or 10 and it is easily seen from the lexicographic array that the second letter must be 1, we conclude with the fact that the set of all semi-Abelian returns is $\{0, 1, 01\}$.

□

Further, the authors of [17] provided a characterization of the words having two semi-Abelian returns and three semi-Abelian returns.

Proposition 4.17. i) A factor v of a Sturmian word has two semi-Abelian returns if and only if v is the only word in its Abelian class.

ii) If a factor of a Sturmian word has three semi-Abelian returns of lengths $l_1 \leq l_2 < l_3$, then $l_3 = l_1 + l_2$.

△

We conclude this section with a condition for finiteness of the sets $SAPR$ in this case.

Theorem 4.18 ([21])**.** Let $\boldsymbol{u}$ be a Sturmian word. The set $SAPR(\boldsymbol{u})$ is finite if and only if $\boldsymbol{u}$ does not have a null intercept (a definition can be found, e.g., in [12]). △

4.2 *d*-bonacci word

For the d-bonacci word, i.e., the unique fixed point of the substitution (2.1), we found semi-Abelian returns of shorter factors by our computer program. For illustration, there are presented some of the results for $d \in \{2, 3, 4, 7\}$ in Appendix C.

The Fibonacci word (i.e., $d = 2$) is a Sturmian word, therefore, it satisfies the characterization from 4.1. Moreover, with respect to the the computer result, we conjecture that the second statement of Proposition 4.17 can be improved in the following way.

Conjecture 4.19. If a factor of the Fibonacci word has

- three semi-Abelian returns w_1, w_2, w_3 of lengths $l_1 \leq l_2 < l_3$ respectively, then $w_3 = w_1 w_2$ or $w_3 = w_2 w_1$;
- two semi-Abelian returns w_1, w_2, then w_1 is a prefix of w_2 or vice versa.

△

Further, all semi-Abelian returns of prefixes for the case of the Fibonacci word are already known.

Theorem 4.20 ([21])**.** Let $\boldsymbol{f}$ be the Fibonacci word. The the set $SAPR(\boldsymbol{f})$, i.e., the set of all semi-Abelian returns to prefixes for the case of $\boldsymbol{f}$ contains exactly the words 0, 1, 01, 10, and 001. △

For $d > 2$, it seems there is no simple characterization of semi-Abelian returns. One could propose to generalize the conjecture, i.e., that the set of all semi-Abelian returns of a factor v in the d-bonacci word consists of d "basic" words and of words created by their concatenation. Nevertheless, we have found a counterexample. For example, for $d = 7$ and $\Psi(v) = (2, 1, 0, 1, 0, 0, 0)$, the set of semi-Abelian returns of v is $\{3010201040102, 1, 0, 301020102, 30102010102, 3010201050102, 301020100102, 3010201060102, 30102\}$

4.3 Thue-Morse word

In the case of the Thue-Morse word, there are exactly 8 semi-Abelian returns to prefixes.

Theorem 4.21 ([21]). Let $\boldsymbol{TM}_{(0)}$ be the Thue-Morse word (as it was defined in Section 3.2).

Then the set $SAPR(\boldsymbol{TM}_{(0)})$ of the semi-Abelian returns to its prefixes contains exactly the words $0, 1, 01, 10, 001, 011, 100, 110, 0011, 0101, 1010, 1100, 00101, 01011, 10100$, and 11010. Δ

4.4 Connection with Abelian complexity

There exists a connection between Abelian return words and Abelian complexity – finiteness of the set $SAPR(\boldsymbol{u})$ of a recurrent word $\boldsymbol{u}$ is a sufficient condition for bounded Abelian complexity.

Theorem 4.22 ([21]). Let $\boldsymbol{u}$ be an infinite Abelian recurrent word, i.e., for every factor v, where $v = \boldsymbol{u}_{[i,i+n)}$ for some integers $i \geq 0$ and $n \geq 1$, there exists an integer i and j, where $j > i$, such that the words v and $\boldsymbol{u}_{[j,j+n)}$ are in the same Abelian class. If the set $SAPR(\boldsymbol{u})$ is finite, then $\boldsymbol{u}$ has bounded Abelian complexity. Δ

CHAPTER 5

Summary

In this work, we have studied balance and Abelian properties of infinite words. We have focused mostly on the d-bonacci word.

We have made a summary of known results about Abelian complexity and Abelian return words. In particular, we have provided a detailed proof of an implication in Theorem 4.3, which is presented in a shorted version in the original article.

Let us mainly mention our original results in this work.

i) Using numerical estimates on the balance function, we have proven that for all $d \in \{2, \ldots, 12\}$, the d-bonacci word is c-balanced with $c = d - 1$, which is a famous conjecture (first time published in [19]). Moreover, we have shown that this bound may be diminished and we have also done it for some d's (see Table 2.3, Table 2.4, Figure 2.1, and the resulting theorems in Chapter 2; detailed tables can be found in Appendix A).

ii) From the obtained balances of the d-bonacci word, we have estimated the upper bounds on the Abelian complexity (see Table 3.2).

iii) For the d-bonacci word, we have examined factors of various lengths using our computer program and we have made a summary of the obtained results (see Table 3.1). Values of the Abelian complexity for lower n's can be found in Appendix B).

iv) We have studied semi-Abelian returns of the d-bonacci word and examined them using our computer program. Obtained tables are presented in Appendix C. We attempted to extend the characterization of semi-Abelian returns, given by Theorem 4.3, for $d > 2$, nevertheless, with respect to the obtained tables, we have concluded with the fact that there probably does not exist such simple characterization for a general d.

Bibliography

[1] B. Adamczewski. Balances for fixed points of primitive substitutions. *Theor. Comput. Sci.* **307**, p. 47–75, 2003. 17, 25

[2] B. Adamczewski. Symbolic discrepancy and self-similar dynamics. *Ann. de l'inst. Four.* **54**, p. 2201–2234, 2004. 17, 25

[3] L. Balková, K. Břinda, and O. Turek. Abelian complexity of infinite words associated with quadratic Parry numbers. *Theor. Comput. Sci.* **412**, p. 6252—6260, 2011. 25, 28

[4] K. Břinda. Abelian complexity of infinite words. Bachelor thesis, *Czech Technical University in Prague*, 2011. 17, 25

[5] K. Břinda. On the balance of d-bonacci word. Extended abstract, *RuFiDiM '12, Proc. of the Sec. Russ. Finn. Symp. on Discr. Math.*, 2012. 23

[6] J. Cassaigne, S. Ferenczi, and L. Q. Zamboni. Imbalances in Arnoux-Rauzy sequences. *Ann. Inst. Fourier* **50**, p. 1265–1276, 2000. 17

[7] J. Cassaigne, G. Richomme, K. Saari, and L. Q. Zamboni. Avoiding Abelian powers in binary words with bounded Abelian complexity. *Int. J. Found. Comput. Sci.* **22**, p. 905—920, 2011. 25

[8] E. M. Coven and G. A. Hedlund. Sequences with minimal block growth. *Math. Syst. Theory* **7**, p. 138–153, 1973. 25

[9] J. Currie and N. Rampersad. Recurrent words with constant Abelian complexity. *Adv. Appl. Math.* **47**, p. 116–124, 2011. 25

[10] J.-M. Dumont and A. Thomas. Systèmes de numération et fonctions fractales relatifs aux substitutions. *Theor. Comput. Sci.* **65**, p. 153–169, 1989. 18

[11] O. Jenkinson and L. Q. Zamboni. Characterisations of balanced words via orderings. *Theor. Comput. Sci.* **310**, p. 247–271, 2004. 30, 31

[12] M. Lothaire. *Algebraic Combinatorics on Words*, volume 90 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2002. 15, 38

[13] B. Madill and N. Rampersad. The abelian complexity of the paperfolding word. Preprint: *arXiv:1208.2856v1 [math.CO]*, 2012. 25, 28

[14] S. Mantaci and A. Restivo. Burrows–Wheeler transform and Sturmian words. *Information Processing Letters*, **86** (5), p. 241–246, 2003. 32

[15] M. Morse and G. A. Hedlund. Symbolic dynamics. *Amer. J. Math.* **60**, p. 815–866, 1938. 17

[16] N. Pytheas Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics*, volume 1794 of *Lecture Notes in Mathematics*, Springer, 2002. 17

[17] S. Puzynina and L. Q. Zamboni. Abelian returns in Sturmian words. Preprint: *arXiv:1204.5755v1 [math.CO]*, 2012. 29, 30, 34, 38

[18] G. Rauzy. Sequences defined by iterated morphisms. *Sequences* (Naples/Positano, 1988), p. 275–286. Springer, 1990. 18

[19] G. Richomme, K. Saari, and L. Q. Zamboni. Balance and Abelian complexity of the Tribonacci word. *Adv. in App. Math.* **45**, p. 212–231, 2010. 17, 25, 27, 41

[20] G. Richomme, K. Saari, and L. Q. Zamboni. Abelian complexity in minimal subshifts. *J. London Math. Soc.* **83**(1), p. 79–95, 2011. 25, 27

[21] M. Rigo, P. Salimov, and E. Vandomme. Some properties of abelian return words. Preprint, 2012. 29, 38, 39

[22] A. Saarela. Ultimately constant abelian complexity of infinite words. *J. Autom. Lang. Comb.* **14**, p. 255–258, 2009. 25

[23] O. Turek. Balances and Abelian complexity of a certain class of infinite ternary words. *RAIRO: Theor. Informatics Appl.* **44**, p. 313–337, 2010. 25, 28

[24] O. Turek. Abelian complexity and Abelian co-decomposition. Preprint: *arXiv:1201.2109v1 [math.CO]*, 2012. 25, 27

[25] S. Widmer. Topics in Word Complexity. Doctoral thesis, *University of Lyon*, 2010. 25, 26

Index

APPENDIX A

Estimates on balance function of d-bonacci word

A.1 Tribonacci word

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
3	0	5	35	8	43	2
3	1	5	21	15	36	2
3	2	5	5	23	28	2

Table A.1: *Lower estimates on maximum of the balance function $B_a(n)$ of the Tribonacci word for the letters $a \in \{0, \dots, d-1\}$ using the factors $\boldsymbol{u}_{[n_1, n_1+n)}$ and $\boldsymbol{u}_{[n_2, n_2+n)}$.*

d	a	i_{max}	$\sum_{i=0}^{i_{max}} \lvert g_{(a,i)} \rvert$	$E_{(a, i_{max}+1)}$	$B_a(n)$ upp. est.
3	0	10	1.20374	0.0650242	2
3	1	10	1.33946	0.0740134	2
3	2	10	1.37527	0.0881861	2

Table A.2: *Upper estimates on the balance function $B_a(n)$ of the Tribonacci word for the letters $a \in \{0, \dots, d-1\}$.*

A.2 4-bonacci word

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
4	0	6	83	16	99	2
4	1	6	53	31	84	2
4	2	6	5	55	60	2
4	3	6	99	8	107	2

Table A.3: *Lower estimates on maximum of the balance function $B_a(n)$ of the 4-bonacci word for the letters $a \in \{0, \dots, d-1\}$ using the factors $\boldsymbol{u}_{[n_1, n_1+n)}$ and $\boldsymbol{u}_{[n_2, n_2+n)}$.*

d	a	i_{max}	$\sum_{i=0}^{i_{max}} \lvert g_{(a,i)} \rvert$	$E_{(a,i_{max}+1)}$	$B_a(n)$ upp. est.
4	0	12	1.27785	0.200545	2
4	1	12	1.51571	0.222133	3
4	2	12	1.56111	0.259158	3
4	3	12	1.57757	0.310565	3

Table A.4: *Upper estimates on the balance function $B_a(n)$ of the 4-bonacci word for the letters $a \in \{0, \dots, d-1\}$.*

A.3 5-bonacci word

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
5	0	21	691183	1228016	1919199	2
5	1	21	2483233	331991	2815224	3
5	2	21	1841859	652678	2494537	3
5	3	21	580951	1283132	1864083	3
5	4	21	1897929	624643	2522572	3

Table A.5: *Lower estimates on maximum of the balance function $B_a(n)$ of the 5-bonacci word for the letters $a \in \{0, \dots, d-1\}$ using the factors $\boldsymbol{u}_{[n_1,n_1+n)}$ and $\boldsymbol{u}_{[n_2,n_2+n)}$.*

d	a	i_{max}	$\sum_{i=0}^{i_{max}} \lvert g_{(a,i)} \rvert$	$E_{(a,i_{max}+1)}$	$B_a(n)$ upp. est.
5	0	32	1.45453	0.0378823	2
5	1	32	1.79242	0.041632	3
5	2	32	1.85452	0.0467633	3
5	3	32	1.87876	0.0538261	3
5	4	32	1.89757	0.0628976	3

Table A.6: *Upper estimates on the balance function $B_a(n)$ of the 5-bonacci word for the letters $a \in \{0, \dots, d-1\}$.*

A.4 6-bonacci word

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
6	0	22	651017	3325799	3976816	2
6	1	22	5883703	709456	6593159	3
6	2	22	4020277	1641169	5661446	3
6	3	22	791825	3255395	4047220	3
6	4	22	5612077	845269	6457346	3
6	5	22	3949291	1676662	5625953	3

Table A.7: *Lower estimates on maximum of the balance function $B_a(n)$ of the 6-bonacci word for the letters $a \in \{0, \dots, d-1\}$ using the factors $\boldsymbol{u}_{[n_1,n_1+n)}$ and $\boldsymbol{u}_{[n_2,n_2+n)}$.*

d	a	i_{max}	$\sum_{i=0}^{i_{max}} \lvert g_{(a,i)} \rvert$	$E_{(a,i_{max}+1)}$	$B_a(n)$ upp. est.
6	0	46	1.5452	0.0441287	3
6	1	46	1.94462	0.0481747	3
6	2	46	2.04221	0.0527744	4
6	3	46	2.06151	0.0591188	4
6	4	46	2.07289	0.0669734	4
6	5	46	2.08192	0.0763591	4

Table A.8: *Upper estimates on the balance function* $B_a(n)$ *of the* 6*-bonacci word for the letters* $a \in \{0, \ldots, d-1\}$.

A.5 7-bonacci word

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
7	0	25	56254263	2970704	59224967	2
7	1	25	48353847	6920912	55274759	3
7	2	25	34623253	13786209	48409462	3
7	3	25	7272401	27461635	34734036	3
7	4	25	47209517	7493077	54702594	3
7	5	25	32343787	14925942	47269729	3
7	6	25	2731785	29731943	32463728	3

Table A.9: *Lower estimates on maximum of the balance function* $B_a(n)$ *of the* 7*-bonacci word for the letters* $a \in \{0, \ldots, d-1\}$ *using the factors* $\boldsymbol{u}_{[n_1,n_1+n)}$ *and* $\boldsymbol{u}_{[n_2,n_2+n)}$.

d	a	i_{max}	$\sum_{i=0}^{i_{max}} \lvert g_{(a,i)} \rvert$	$E_{(a,i_{max}+1)}$	$B_a(n)$ upp. est.
7	0	50	1.61484	0.136779	3
7	1	50	2.06787	0.148962	4
7	2	50	2.1871	0.160787	4
7	3	50	2.21446	0.176272	4
7	4	50	2.22033	0.195169	4
7	5	50	2.22283	0.217483	4
7	6	50	2.22619	0.243101	4

Table A.10: *Upper estimates on the balance function* $B_a(n)$ *of the* 7*-bonacci word for the letters* $a \in \{0, \ldots, d-1\}$.

A.6 8-bonacci word

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
8	0	77	2338145807974257 27268323	1461060084864116 9146990	2484251816460668 96415313	3

Table A.11: *Lower estimates on maximum of the balance function* $B_a(n)$ *of the* 8*-bonacci word for the letters* $a \in \{0, \ldots, d-1\}$ *using the factors* $\boldsymbol{u}_{[n_1,n_1+n)}$ *and* $\boldsymbol{u}_{[n_2,n_2+n)}$.

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
8	1	77	200819785059652345360603	31107998717527860100850	231927783777180205461453	3
8	2	77	138850728146990638227277	62092527173858713667513	200943255320849351894790	4
8	3	77	15158541960556965633891	123938620267075549964206	139097162227632515598097	4
8	4	77	231734918358232683363929	15650432068237691099187	247385350426470374463116	4
8	5	77	200558281725641179368615	31238750384533443096844	231797032110174622465459	4
8	6	77	138328742927707216382487	62353519783500424589908	200682262711207640972395	4
8	7	77	14116643186476334729665	124459569654115865416319	138576212840592200145984	4

Table A.11: *Lower estimates on maximum of the balance function $B_a(n)$ of the 8-bonacci word for the letters $a \in \{0, \dots, d-1\}$ using the factors $\boldsymbol{u}_{[n_1,n_1+n)}$ and $\boldsymbol{u}_{[n_2,n_2+n)}$.*

d	a	i_{max}	$\sum_{i=0}^{i_{max}} \lvert g_{(a,i)} \rvert$	$E_{(a,i_{max}+1)}$	$B_a(n)$ upp. est.
8	0	103	1.72816	0.0226902	3
8	1	103	2.2528	0.0246892	4
8	2	103	2.40544	0.0263863	4
8	3	103	2.43739	0.0284768	4
8	4	103	2.44319	0.031025	4
8	5	103	2.44657	0.0339742	4
8	6	103	2.45038	0.037327	4
8	7	103	2.45375	0.0410838	4

Table A.12: *Upper estimates on the balance function $B_a(n)$ of the 8-bonacci word for the letters $a \in \{0, \dots, d-1\}$.*

A.7 9-bonacci word

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
9	0	87	252568808723018474566558829	16871200439392572679803081	269440009162411047246361910	3
9	1	87	252502579843928543396609419	16904314878937538264777786	269406894722866081661387205	4
9	2	87	218760570996129074109596829	33775319302837272908284081	252535890298966347017880910	4
9	3	87	151343042932628534163381973	67484083334587542881391509	218827126267216077044773482	4

Table A.13: *Lower estimates on maximum of the balance function $B_a(n)$ of the 9-bonacci word for the letters $a \in \{0, \dots, d-1\}$ using the factors $\boldsymbol{u}_{[n_1,n_1+n)}$ and $\boldsymbol{u}_{[n_2,n_2+n)}$.*

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
9	4	87	16605469122479276467851083	134852870239662171729156954	151458339362141448197008037	4
9	5	87	251442662154942292044676663	17434273723430663940744164	268876935878372955985420827	4
9	6	87	218892897153227297800258457	33709156224288161062953267	252602053377515458863211724	4
9	7	87	151325342392197530188303301	67492933604803044868930845	218818275997000575057234146	4
9	8	87	16605468848098988061973317	134852870376852315932095837	151458339224951303994069154	4

Table A.13: *Lower estimates on maximum of the balance function $B_a(n)$ of the 9-bonacci word for the letters $a \in \{0,\ldots,d-1\}$ using the factors $\boldsymbol{u}_{[n_1,n_1+n)}$ and $\boldsymbol{u}_{[n_2,n_2+n)}$.*

d	a	i_{max}	$\sum_{i=0}^{i_{max}} \lvert g_{(a,i)} \rvert$	$E_{(a,i_{max}+1)}$	$B_a(n)$ upp. est.
9	0	104	1.8079	0.0940223	3
9	1	104	2.38607	0.102348	4
9	2	104	2.56452	0.108601	5
9	3	104	2.60612	0.115963	5
9	4	104	2.61222	0.12474	5
9	5	104	2.61271	0.134805	5
9	6	104	2.61431	0.146127	5
9	7	104	2.61524	0.158702	5
9	8	104	2.61552	0.17257	5

Table A.14: *Upper estimates on the balance function $B_a(n)$ of the 9-bonacci word for the letters $a \in \{0,\ldots,d-1\}$.*

A.8 10-bonacci word

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
10	0	83	17518846531354920546923511	567148898747730217348100	18085995430102650764271611	3
10	1	83	16385661896338241681890441	1133741216256069649864635	17519403112594311331755076	4
10	2	83	14124983057517139817930225	2264080635666620581844743	16389063693183760399774968	4
10	3	83	9601265593160092699789013	4525939367845144140915349	14127204961005236840704362	4
10	4	83	557696901598097032712679	9047723713626141974453516	9605420615224239007166195	4

Table A.15: *Lower estimates on maximum of the balance function $B_a(n)$ of the 10-bonacci word for the letters $a \in \{0,\ldots,d-1\}$ using the factors $\boldsymbol{u}_{[n_1,n_1+n)}$ and $\boldsymbol{u}_{[n_2,n_2+n)}$.*

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
10	5	83	17519992245452997019515689	566576041698691981052011	18086568287151689000567700	4
10	6	83	16387952200444739996616491	1132596064202820492501610	17520548264647560489118101	4
10	7	83	14124983057499252689023203	2264080635675564146298254	16389063693174816835321457	4
10	8	83	9601265575503215235658955	4525939376673582872980378	14127204952176798108639333	4
10	9	83	558270039876714881257035	9047437144486833050181338	9605707184363547931438373	4

Table A.15: *Lower estimates on maximum of the balance function* $B_a(n)$ *of the* 10*-bonacci word for the letters* $a \in \{0, \dots, d-1\}$ *using the factors* $\boldsymbol{u}_{[n_1,n_1+n)}$ *and* $\boldsymbol{u}_{[n_2,n_2+n)}$.

d	a	i_{max}	$\sum_{i=0}^{i_{max}} \lvert g_{(a,i)} \rvert$	$E_{(a,i_{max}+1)}$	$B_a(n)$ upp. est.
10	0	143	1.90734	0.0791713	3
10	1	143	2.54608	0.0862745	5
10	2	143	2.74995	0.0910962	5
10	3	143	2.80347	0.0964823	5
10	4	143	2.81301	0.102767	5
10	5	143	2.81325	0.10994	5
10	6	143	2.81299	0.117925	5
10	7	143	2.81306	0.126718	5
10	8	143	2.81336	0.136339	5
10	9	143	2.81385	0.146828	5

Table A.16: *Upper estimates on the balance function* $B_a(n)$ *of the* 10*-bonacci word for the letters* $a \in \{0, \dots, d-1\}$.

A.9 11-bonacci word

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
11	0	71	4370227657815678675480	141077396217804843764	4511305054033483519244	3
11	1	71	4379189619946574092608	136596415152357135200	4515786035098931227808	4
11	2	71	4087926004180809573886	282228223035239394561	4370154227216048968447	4
11	3	71	3523745914869670893368	564318267690808734820	4088064182560479628188	4
11	4	71	2395661957703010980586	1128360246274138691211	3524022203977149671797	4

Table A.17: *Lower estimates on maximum of the balance function* $B_a(n)$ *of the* 11*-bonacci word for the letters* $a \in \{0, \dots, d-1\}$ *using the factors* $\boldsymbol{u}_{[n_1,n_1+n)}$ *and* $\boldsymbol{u}_{[n_2,n_2+n)}$.

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
11	5	71	1400463510434478 26126	2256168049603920 268441	2396214400647368 094567	4
11	6	71	4370080517336909 589594	1411509664571893 86707	4511231483794098 976301	4
11	7	71	4087916798890411 877936	2822328256804382 42536	4370149624570850 120472	4
11	8	71	3523727508795767 469486	5643274707277604 46761	4088054979523527 916247	4
11	9	71	2395625154566781 503468	1128378647842253 429770	3524003802409034 933238	4
11	10	71	1399727627897315 62730	2256204843730778 400139	2396177606520509 962869	4

Table A.17: *Lower estimates on maximum of the balance function $B_a(n)$ of the 11-bonacci word for the letters $a \in \{0, \dots, d-1\}$ using the factors $\boldsymbol{u}_{[n_1,n_1+n)}$ and $\boldsymbol{u}_{[n_2,n_2+n)}$.*

d	a	i_{max}	$\sum_{i=0}^{i_{max}} \lvert g_{(a,i)}\rvert$	$E_{(a,i_{max}+1)}$	$B_a(n)$ upp. est.
11	0	220	2.01378	0.0327594	4
11	1	220	2.71601	0.0357528	5
11	2	220	2.94927	0.0376171	5
11	3	220	3.01448	0.0395933	6
11	4	220	3.0272	0.0418555	6
11	5	220	3.02704	0.044412	6
11	6	220	3.02591	0.0472392	6
11	7	220	3.02578	0.0503306	6
11	8	220	3.02621	0.0536884	6
11	9	220	3.02674	0.0573238	6
11	10	220	3.02719	0.0612537	6

Table A.18: *Upper estimates on the balance function $B_a(n)$ of the 11-bonacci word for the letters $a \in \{0, \dots, d-1\}$.*

A.10 12-bonacci word

d	a	i_{max}	n	n_1	n_2	max $B_a(n)$ low. est.
12	0	212	1205340629993642 3563203602585813 8051421668275677 0363647322896761 9	3955494551510977 2891445952242891 8857307492883698 526851196377198	1244895575508752 1292118062108242 7239994743204514 0216332442534481 7	3

Table A.19: *Lower estimates on maximum of the balance function $B_a(n)$ of the 12-bonacci word for the letters $a \in \{0, \dots, d-1\}$ using the factors $\boldsymbol{u}_{[n_1,n_1+n)}$ and $\boldsymbol{u}_{[n_2,n_2+n)}$.*

d	a	i_{max}	n	n_1	n_2	**max $B_a(n)$ low. est.**
12	1	212	11262893659236276392122804883695659093900312776359953527665900117	7908057755011713144548583734879918814407502793703683239779 10949	12053699434737447706577663257183650975341063055730321851643811066	4
12	2	212	11287624562728067539748942924801248549197702842255405346736083803	778440323755275740641789352935197153792055246422642414442819106	12066064886483343280390732277736445702989758088678047761178902909	5
12	3	212	9731100018155064151533538889459969682597853729769263000906780571	1556702596041777434749491370605836587091979802665713587357470722	11287802614196841586283030260065806269689833532434976588264251293	5
12	4	212	6618456050722724425815412189973558760686148785248670739290514311	3113024579757947297608554720349042048047832274926009718165603852	9731480630480671723423966910322600808733981060174680457456118163	5
12	5	212	393929154390546405272310749842311389950227359592651936537548027	6225288027924036307880105440414665733415792987754019119542086994	6619217182314582713152416190256977123366020347346671056079635021	5
12	6	212	12053602747282014635864825522629943285438093864361946497869066605	395451231478302192583848054020849785671859735369371838876327705	12449053978760316828448673576650793071109953599731318336745394310	5
12	7	212	11262893659236276392122890674940166672460916378077390627874128131	790805775501171314454815477865738092160448478511649773873796942	12053699434737447706577706152805904764621364856589040401747925073	5
12	8	212	9681668810775681592988315777429417266827098881109454325053729105	1581418199731468714022102926621112794977357226995617925283996455	11263087010507150307010418704050530061804456108105072250337725560	5
12	9	212	6519605721847830830223707749062216433132914089465406968342230691	3162449744195394095404406940804713211824449622817641603639745662	9682055466043224925628114689866929644957363712283048571981976353	5

Table A.19: *Lower estimates on maximum of the balance function $B_a(n)$ of the 12-bonacci word for the letters $a \in \{0, \dots, d-1\}$ using the factors $\boldsymbol{u}_{[n_1, n_1+n)}$ and $\boldsymbol{u}_{[n_2, n_2+n)}$.*

$\boldsymbol{d}$	$\boldsymbol{a}$	$\boldsymbol{i_{max}}$	$\boldsymbol{n}$	$\boldsymbol{n_1}$	$\boldsymbol{n_2}$	**max** $\boldsymbol{B_a(n)}$ **low. est.**
12	10	212	1962526654535149 7881027825764380 2563414528622394 152411154701115	6324126272392552 0211111216865139 2014668364235635 3268882233510450	6520378937846066 9999213999441577 2271009817097874 7421293388211565	5
12	11	212	1244890739343981 8024278574969570 6048342408237618 8724512026814176 5	1977989083994004 9837697333055051 9011270494786606 722527676790125	1264670630183921 8522655548300121 1238455113185484 9396764794493189 0	5

Table A.19: *Lower estimates on maximum of the balance function $B_a(n)$ of the 12-bonacci word for the letters $a \in \{0, \dots, d-1\}$ using the factors $\boldsymbol{u}_{[n_1,n_1+n)}$ and $\boldsymbol{u}_{[n_2,n_2+n)}$.*

$\boldsymbol{d}$	$\boldsymbol{a}$	$\boldsymbol{i_{max}}$	$\sum_{\boldsymbol{i=0}}^{\boldsymbol{i_{max}}} \lvert \boldsymbol{g_{(a,i)}} \rvert$	$\boldsymbol{E_{(a,i_{max}+1)}}$	$\boldsymbol{B_a(n)}$ **upp. est.**
12	0	218	2.10485	0.116237	4
12	1	218	2.85967	0.127082	5
12	2	218	3.1194	0.133365	6
12	3	218	3.19648	0.139688	6
12	4	218	3.2137	0.146786	6
12	5	218	3.21419	0.154724	6
12	6	218	3.2124	0.16346	6
12	7	218	3.21142	0.172952	6
12	8	218	3.21079	0.183199	6
12	9	218	3.21027	0.194225	6
12	10	218	3.20983	0.20607	6
12	11	218	3.20958	0.218788	6

Table A.20: *Upper estimates on the balance function $B_a(n)$ of the 12-bonacci word for the letters $a \in \{0, \dots, d-1\}$.*

APPENDIX B

Abelian complexity of d-bonacci word

B.1 Tribonacci word

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
1	3	2	3	3	4	4	3	5	4	6	4	7	4	8	3
9	4	10	4	11	4	12	4	13	4	14	4	15	3	16	4
17	4	18	4	19	4	20	4	21	4	22	4	23	4	24	4
25	4	26	4	27	4	28	3	29	4	30	5	31	5	32	4
33	4	34	4	35	4	36	4	37	5	38	5	39	4	40	4
41	4	42	4	43	4	44	4	45	4	46	4	47	4	48	4
49	4	50	4	51	4	52	3	53	4	54	4	55	5	56	4
57	5	58	4	59	4	60	4	61	4	62	4	63	4	64	4
65	4	66	4	67	4	68	5	69	4	70	5	71	4	72	4
73	4	74	5	75	5	76	4	77	4	78	4	79	4	80	4
81	5	82	5	83	4	84	4	85	4	86	4	87	4	88	4
89	4	90	4	91	4	92	4	93	4	94	4	95	4	96	3
97	4	98	4	99	4	100	4	101	5	102	4	103	4	104	4
105	5	106	4	107	4	108	4	109	4	110	4	111	5	112	5
113	4	114	4	115	4	116	4	117	4	118	5	119	5	120	4
121	4	122	4	123	4	124	4	125	5	126	4	127	4	128	4
129	5	130	4	131	4	132	4	133	4	134	4	135	4	136	5
137	4	138	5	139	4	140	4	141	4	142	4	143	4	144	4
145	4	146	4	147	4	148	4	149	5	150	4	151	5	152	4
153	4	154	4	155	5	156	5	157	4	158	4	159	4	160	4
161	4	162	5	163	5	164	4	165	4	166	4	167	4	168	4
169	4	170	4	171	4	172	4	173	4	174	4	175	4	176	4
177	3	178	4	179	5	180	5	181	4	182	4	183	4	184	4
185	5	186	5	187	5	188	4	189	4	190	4	191	4	192	5
193	5	194	5	195	4	196	4	197	4	198	4	199	5	200	5
201	4	202	4	203	4	204	5	205	4	206	5	207	4	208	4

Table B.1: *The Abelian complexity $\mathcal{AC}(n)$ of the Tribonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
209	4	210	4	211	4	212	4	213	4	214	4	215	4	216	4
217	5	218	4	219	5	220	4	221	4	222	4	223	5	224	5
225	4	226	4	227	4	228	4	229	5	230	5	231	5	232	4
233	4	234	4	235	4	236	5	237	5	238	5	239	4	240	4
241	4	242	4	243	5	244	5	245	4	246	4	247	4	248	4
249	4	250	5	251	4	252	4	253	4	254	5	255	4	256	4
257	4	258	4	259	4	260	5	261	5	262	4	263	4	264	4
265	4	266	4	267	5	268	5	269	4	270	4	271	4	272	4
273	4	274	5	275	4	276	4	277	4	278	5	279	4	280	4
281	4	282	4	283	4	284	4	285	5	286	4	287	5	288	4
289	4	290	4	291	4	292	4	293	4	294	4	295	4	296	4
297	4	298	5	299	4	300	5	301	4	302	4	303	4	304	5
305	5	306	4	307	4	308	4	309	4	310	4	311	5	312	5
313	4	314	4	315	4	316	4	317	4	318	4	319	4	320	4
321	4	322	4	323	4	324	4	325	4	326	3	327	4	328	4
329	5	330	4	331	5	332	4	333	4	334	4	335	4	336	4
337	4	338	4	339	4	340	5	341	5	342	6	343	4	344	5
345	4	346	4	347	4	348	5	349	5	350	4	351	4	352	4
353	5	354	4	355	6	356	5	357	5	358	4	359	4	360	4
361	4	362	4	363	4	364	4	365	4	366	5	367	4	368	5
369	4	370	4	371	4	372	4	373	4	374	4	375	5	376	4
377	4	378	4	379	5	380	4	381	4	382	4	383	4	384	4
385	5	386	5	387	4	388	4	389	4	390	4	391	4	392	5
393	5	394	4	395	4	396	4	397	4	398	4	399	5	400	4
401	4	402	4	403	5	404	4	405	4	406	4	407	4	408	4
409	4	410	5	411	4	412	5	413	4	414	4	415	4	416	4
417	4	418	4	419	4	420	4	421	5	422	5	423	6	424	4
425	5	426	4	427	4	428	4	429	5	430	5	431	4	432	4
433	4	434	5	435	4	436	6	437	5	438	5	439	4	440	4
441	4	442	4	443	4	444	4	445	4	446	4	447	5	448	4
449	5	450	4	451	4	452	4	453	5	454	5	455	4	456	4
457	4	458	4	459	5	460	5	461	5	462	4	463	4	464	4
465	4	466	5	467	5	468	5	469	4	470	4	471	4	472	4
473	5	474	5	475	4	476	4	477	4	478	5	479	4	480	5
481	4	482	4	483	4	484	4	485	4	486	4	487	4	488	4
489	4	490	4	491	5	492	4	493	5	494	4	495	4	496	4
497	5	498	5	499	4	500	4	501	4	502	4	503	5	504	5
505	5	506	4	507	4	508	4	509	4	510	5	511	5	512	5
513	4	514	4	515	4	516	4	517	5	518	5	519	4	520	4
521	4	522	4	523	4	524	5	525	4	526	4	527	4	528	5
529	4	530	4	531	4	532	4	533	4	534	5	535	5	536	4
537	4	538	4	539	4	540	4	541	5	542	5	543	4	544	4
545	4	546	4	547	4	548	5	549	4	550	4	551	4	552	5

Table B.1: *The Abelian complexity $\mathcal{AC}(n)$ of the Tribonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
553	4	554	4	555	4	556	4	557	4	558	4	559	5	560	4
561	5	562	4	563	4	564	4	565	4	566	4	567	4	568	4
569	4	570	4	571	4	572	5	573	4	574	5	575	4	576	4
577	4	578	5	579	5	580	4	581	4	582	4	583	4	584	4
585	5	586	5	587	4	588	4	589	4	590	4	591	4	592	4
593	4	594	4	595	4	596	4	597	4	598	4	599	4	600	3
601	4	602	4	603	4	604	4	605	5	606	4	607	4	608	4
609	5	610	4	611	4	612	4	613	4	614	4	615	5	616	5
617	4	618	4	619	4	620	4	621	4	622	5	623	5	624	4
625	5	626	4	627	5	628	4	629	6	630	4	631	4	632	4
633	5	634	4	635	4	636	4	637	4	638	4	639	4	640	5
641	4	642	5	643	4	644	4	645	4	646	4	647	4	648	4
649	5	650	4	651	4	652	4	653	6	654	4	655	5	656	4
657	5	658	4	659	5	660	5	661	4	662	4	663	4	664	4
665	4	666	5	667	5	668	4	669	4	670	4	671	4	672	4
673	5	674	4	675	4	676	4	677	5	678	4	679	4	680	4
681	4	682	4	683	5	684	5	685	4	686	4	687	4	688	4
689	5	690	5	691	5	692	4	693	4	694	4	695	4	696	5
697	5	698	5	699	4	700	4	701	4	702	4	703	5	704	5
705	4	706	4	707	4	708	5	709	4	710	5	711	4	712	4
713	4	714	4	715	4	716	4	717	4	718	4	719	4	720	4
721	5	722	4	723	5	724	4	725	4	726	4	727	5	728	5
729	4	730	4	731	4	732	4	733	5	734	5	735	5	736	4
737	4	738	4	739	4	740	5	741	5	742	5	743	4	744	4
745	4	746	4	747	5	748	5	749	4	750	4	751	4	752	4
753	4	754	5	755	4	756	4	757	4	758	5	759	4	760	4
761	4	762	4	763	4	764	5	765	5	766	4	767	4	768	4
769	4	770	4	771	5	772	5	773	4	774	5	775	4	776	5
777	4	778	6	779	4	780	4	781	4	782	5	783	4	784	4
785	4	786	4	787	4	788	4	789	5	790	4	791	5	792	4
793	4	794	4	795	4	796	4	797	4	798	5	799	4	800	4
801	4	802	6	803	4	804	5	805	4	806	5	807	4	808	5
809	5	810	4	811	4	812	4	813	4	814	4	815	5	816	5
817	4	818	4	819	4	820	4	821	4	822	5	823	4	824	4
825	4	826	5	827	4	828	4	829	4	830	4	831	4	832	4
833	5	834	4	835	5	836	4	837	4	838	4	839	4	840	4
841	4	842	4	843	4	844	5	845	5	846	6	847	4	848	5
849	4	850	4	851	4	852	5	853	5	854	4	855	4	856	4
857	5	858	4	859	6	860	5	861	5	862	4	863	4	864	4
865	4	866	4	867	4	868	4	869	4	870	5	871	4	872	5
873	4	874	4	875	4	876	4	877	4	878	4	879	5	880	4
881	4	882	4	883	5	884	4	885	4	886	4	887	4	888	4
889	5	890	5	891	4	892	4	893	4	894	4	895	4	896	5

Table B.1: *The Abelian complexity $\mathcal{AC}(n)$ of the Tribonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
897	5	898	4	899	4	900	4	901	4	902	4	903	5	904	4
905	4	906	4	907	5	908	4	909	4	910	4	911	4	912	4
913	4	914	5	915	4	916	5	917	4	918	4	919	4	920	4
921	4	922	4	923	4	924	4	925	5	926	5	927	6	928	4
929	5	930	4	931	4	932	4	933	5	934	5	935	4	936	4
937	4	938	5	939	4	940	6	941	5	942	5	943	4	944	4
945	4	946	4	947	4	948	4	949	4	950	4	951	5	952	4
953	5	954	4	955	4	956	4	957	5	958	5	959	4	960	4
961	4	962	4	963	5	964	5	965	5	966	4	967	4	968	4
969	4	970	5	971	5	972	5	973	4	974	4	975	4	976	4
977	5	978	5	979	4	980	4	981	4	982	5	983	4	984	5
985	4	986	4	987	4	988	4	989	4	990	4	991	4	992	4
993	4	994	4	995	5	996	4	997	5	998	4	999	4	1000	4

Table B.1: *The Abelian complexity $\mathcal{AC}(n)$ of the Tribonacci word for $n \in \{1, \dots, 1000\}$.*

B.2 4-bonacci word

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
1	4	2	4	3	6	4	4	5	7	6	6	7	7	8	4
9	7	10	7	11	8	12	6	13	8	14	7	15	7	16	4
17	7	18	7	19	8	20	7	21	8	22	8	23	7	24	7
25	8	26	8	27	7	28	8	29	7	30	7	31	4	32	7
33	7	34	8	35	7	36	8	37	8	38	8	39	7	40	8
41	8	42	8	43	8	44	7	45	7	46	7	47	7	48	8
49	8	50	8	51	8	52	7	53	8	54	8	55	8	56	7
57	8	58	7	59	7	60	4	61	7	62	8	63	9	64	7
65	8	66	9	67	9	68	7	69	8	70	10	71	10	72	8
73	8	74	8	75	8	76	7	77	9	78	10	79	9	80	8
81	9	82	9	83	8	84	8	85	9	86	10	87	7	88	8
89	7	90	8	91	7	92	8	93	9	94	9	95	8	96	8
97	8	98	8	99	8	100	7	101	8	102	8	103	8	104	8
105	8	106	8	107	8	108	7	109	8	110	8	111	8	112	7
113	8	114	7	115	7	116	4	117	7	118	8	119	10	120	8
121	10	122	10	123	10	124	7	125	8	126	9	127	10	128	8
129	9	130	8	131	8	132	7	133	9	134	10	135	10	136	10
137	11	138	10	139	8	140	8	141	9	142	9	143	8	144	8
145	8	146	8	147	7	148	9	149	9	150	10	151	10	152	9
153	9	154	8	155	8	156	9	157	8	158	9	159	8	160	8
161	8	162	8	163	9	164	9	165	10	166	10	167	9	168	7
169	8	170	9	171	9	172	7	173	8	174	9	175	9	176	7

Table B.2: *The Abelian complexity $\mathcal{AC}(n)$ of the 4-bonacci word for $n \in \{1, \dots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
177	8	178	11	179	11	180	10	181	9	182	10	183	9	184	8
185	9	186	10	187	9	188	8	189	9	190	9	191	8	192	8
193	10	194	11	195	9	196	9	197	9	198	9	199	8	200	8
201	9	202	9	203	8	204	8	205	8	206	8	207	8	208	7
209	8	210	8	211	8	212	8	213	8	214	8	215	8	216	7
217	8	218	8	219	8	220	7	221	8	222	7	223	7	224	4
225	7	226	7	227	9	228	7	229	10	230	9	231	10	232	7
233	10	234	9	235	11	236	8	237	10	238	8	239	8	240	7
241	8	242	8	243	9	244	9	245	10	246	9	247	8	248	9
249	9	250	9	251	8	252	8	253	8	254	8	255	7	256	9
257	8	258	10	259	9	260	10	261	9	262	10	263	9	264	11
265	9	266	10	267	8	268	8	269	8	270	9	271	9	272	9
273	8	274	10	275	9	276	8	277	8	278	10	279	10	280	8
281	8	282	8	283	8	284	7	285	10	286	11	287	11	288	8
289	11	290	10	291	11	292	8	293	11	294	11	295	10	296	8
297	8	298	8	299	8	300	9	301	10	302	10	303	8	304	9
305	9	306	9	307	8	308	8	309	9	310	9	311	8	312	8
313	8	314	9	315	9	316	9	317	9	318	10	319	9	320	10
321	9	322	9	323	8	324	7	325	8	326	9	327	10	328	9
329	10	330	10	331	10	332	7	333	8	334	9	335	10	336	8
337	9	338	8	339	8	340	7	341	9	342	10	343	11	344	10
345	12	346	10	347	10	348	8	349	10	350	9	351	10	352	8
353	9	354	8	355	8	356	9	357	9	358	10	359	10	360	9
361	9	362	8	363	8	364	9	365	8	366	9	367	8	368	8
369	8	370	8	371	9	372	10	373	10	374	11	375	9	376	9
377	8	378	10	379	9	380	9	381	8	382	10	383	9	384	8
385	8	386	11	387	11	388	10	389	9	390	10	391	9	392	8
393	9	394	10	395	9	396	8	397	9	398	9	399	8	400	8
401	10	402	11	403	9	404	9	405	9	406	9	407	8	408	8
409	9	410	9	411	8	412	8	413	8	414	8	415	8	416	7
417	8	418	8	419	8	420	8	421	8	422	8	423	8	424	7
425	8	426	8	427	8	428	7	429	8	430	7	431	7	432	4
433	7	434	7	435	8	436	7	437	9	438	9	439	9	440	7
441	10	442	9	443	10	444	8	445	10	446	8	447	8	448	7
449	10	450	8	451	9	452	9	453	11	454	9	455	8	456	9
457	10	458	9	459	8	460	8	461	8	462	8	463	8	464	9
465	8	466	8	467	9	468	9	469	9	470	9	471	10	472	11
473	9	474	9	475	8	476	8	477	8	478	10	479	10	480	10
481	8	482	10	483	9	484	8	485	8	486	10	487	10	488	8
489	8	490	8	491	8	492	7	493	9	494	10	495	10	496	8
497	10	498	9	499	10	500	8	501	11	502	10	503	10	504	8
505	10	506	8	507	9	508	9	509	12	510	10	511	9	512	9
513	10	514	9	515	8	516	8	517	9	518	9	519	9	520	10

Table B.2: *The Abelian complexity $\mathcal{AC}(n)$ of the 4-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
521	9	522	10	523	10	524	10	525	9	526	8	527	9	528	10
529	9	530	9	531	8	532	8	533	8	534	9	535	10	536	10
537	10	538	11	539	10	540	8	541	8	542	9	543	9	544	8
545	8	546	8	547	8	548	7	549	10	550	9	551	11	552	10
553	11	554	9	555	8	556	8	557	11	558	8	559	10	560	8
561	11	562	8	563	8	564	9	565	11	566	10	567	11	568	9
569	10	570	8	571	9	572	9	573	8	574	8	575	9	576	9
577	8	578	9	579	11	580	12	581	10	582	10	583	10	584	9
585	8	586	10	587	10	588	10	589	8	590	10	591	9	592	8
593	8	594	11	595	11	596	10	597	9	598	10	599	9	600	8
601	8	602	9	603	9	604	8	605	9	606	8	607	9	608	8
609	10	610	9	611	9	612	8	613	10	614	8	615	9	616	8
617	11	618	9	619	9	620	8	621	9	622	8	623	8	624	7
625	8	626	8	627	8	628	9	629	8	630	10	631	9	632	10
633	8	634	10	635	9	636	11	637	8	638	10	639	8	640	8
641	7	642	8	643	8	644	9	645	9	646	10	647	9	648	8
649	9	650	9	651	9	652	8	653	8	654	8	655	8	656	7
657	9	658	8	659	10	660	9	661	11	662	9	663	10	664	9
665	12	666	9	667	10	668	8	669	10	670	8	671	9	672	9
673	10	674	8	675	10	676	9	677	10	678	8	679	10	680	10
681	9	682	8	683	8	684	8	685	8	686	10	687	11	688	11
689	8	690	11	691	10	692	11	693	8	694	11	695	11	696	10
697	8	698	8	699	8	700	8	701	9	702	10	703	10	704	8
705	9	706	9	707	9	708	8	709	8	710	9	711	9	712	8
713	8	714	8	715	9	716	9	717	10	718	9	719	10	720	9
721	11	722	9	723	9	724	8	725	9	726	8	727	9	728	10
729	10	730	10	731	10	732	10	733	9	734	8	735	9	736	10
737	9	738	9	739	8	740	8	741	8	742	9	743	10	744	11
745	10	746	12	747	10	748	10	749	8	750	10	751	9	752	10
753	8	754	9	755	8	756	8	757	9	758	9	759	10	760	10
761	9	762	9	763	8	764	8	765	9	766	8	767	9	768	8
769	8	770	8	771	8	772	9	773	10	774	10	775	11	776	9
777	9	778	8	779	10	780	9	781	9	782	8	783	10	784	9
785	8	786	8	787	11	788	11	789	10	790	9	791	10	792	9
793	8	794	9	795	10	796	9	797	8	798	9	799	9	800	8
801	8	802	10	803	11	804	9	805	9	806	9	807	9	808	8
809	8	810	9	811	9	812	8	813	8	814	8	815	8	816	8
817	7	818	8	819	8	820	8	821	8	822	8	823	8	824	8
825	7	826	8	827	8	828	8	829	7	830	8	831	7	832	7
833	4	834	7	835	8	836	9	837	7	838	8	839	9	840	9
841	7	842	9	843	10	844	11	845	8	846	9	847	8	848	8
849	8	850	10	851	10	852	10	853	9	854	10	855	9	856	8
857	10	858	10	859	10	860	8	861	8	862	8	863	8	864	8

Table B.2: *The Abelian complexity $\mathcal{AC}(n)$ of the 4-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
865	10	866	10	867	9	868	9	869	9	870	9	871	9	872	10
873	11	874	10	875	9	876	8	877	8	878	8	879	10	880	10
881	10	882	9	883	10	884	9	885	8	886	8	887	10	888	10
889	8	890	8	891	9	892	10	893	8	894	10	895	11	896	11
897	8	898	8	899	9	900	10	901	8	902	10	903	10	904	10
905	9	906	9	907	10	908	10	909	11	910	12	911	11	912	9
913	9	914	9	915	9	916	8	917	8	918	9	919	9	920	9
921	10	922	10	923	10	924	11	925	10	926	10	927	8	928	9
929	10	930	9	931	9	932	8	933	8	934	8	935	9	936	10
937	10	938	10	939	11	940	10	941	8	942	8	943	10	944	10
945	8	946	8	947	9	948	9	949	7	950	9	951	11	952	12
953	10	954	10	955	10	956	9	957	9	958	10	959	10	960	10
961	9	962	10	963	9	964	8	965	11	966	11	967	12	968	10
969	10	970	10	971	9	972	9	973	10	974	10	975	9	976	9
977	9	978	9	979	9	980	11	981	12	982	12	983	10	984	10
985	9	986	9	987	10	988	10	989	10	990	9	991	10	992	9
993	8	994	8	995	11	996	11	997	10	998	9	999	10	1000	10

Table B.2: *The Abelian complexity $\mathcal{AC}(n)$ of the 4-bonacci word for $n \in \{1, \ldots, 1000\}$.*

B.3 5-bonacci word

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
1	5	2	5	3	8	4	5	5	10	6	8	7	10	8	5
9	11	10	10	11	13	12	8	13	13	14	10	15	11	16	5
17	11	18	11	19	14	20	10	21	15	22	13	23	14	24	8
25	14	26	13	27	15	28	10	29	14	30	11	31	11	32	5
33	11	34	11	35	14	36	11	37	15	38	14	39	14	40	11
41	15	42	15	43	15	44	14	45	15	46	14	47	11	48	11
49	14	50	15	51	14	52	15	53	15	54	15	55	11	56	14
57	14	58	15	59	11	60	14	61	11	62	11	63	5	64	11
65	11	66	14	67	11	68	15	69	14	70	15	71	11	72	15
73	15	74	16	75	14	76	15	77	14	78	14	79	11	80	15
81	15	82	16	83	15	84	15	85	15	86	15	87	14	88	15
89	15	90	14	91	14	92	11	93	11	94	11	95	11	96	14
97	14	98	15	99	15	100	14	101	15	102	15	103	15	104	15
105	16	106	15	107	15	108	11	109	14	110	14	111	15	112	14
113	16	114	15	115	15	116	11	117	15	118	14	119	15	120	11
121	14	122	11	123	11	124	5	125	11	126	12	127	15	128	11
129	15	130	15	131	16	132	11	133	15	134	17	135	18	136	14
137	16	138	16	139	16	140	11	141	15	142	18	143	19	144	15

Table B.3: *The Abelian complexity $\mathcal{AC}(n)$ of the 5-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
145	16	146	18	147	18	148	14	149	15	150	18	151	17	152	14
153	14	154	14	155	14	156	11	157	16	158	18	159	18	160	15
161	17	162	18	163	17	164	15	165	17	166	20	167	17	168	16
169	16	170	17	171	15	172	15	173	18	174	20	175	16	176	16
177	16	178	18	179	14	180	15	181	14	182	17	183	11	184	14
185	11	186	14	187	11	188	15	189	16	190	18	191	14	192	16
193	16	194	17	195	15	196	15	197	17	198	18	199	15	200	16
201	15	202	16	203	15	204	15	205	17	206	17	207	16	208	15
209	16	210	15	211	15	212	11	213	14	214	14	215	14	216	14
217	14	218	15	219	15	220	14	221	16	222	16	223	16	224	15
225	16	226	15	227	15	228	11	229	15	230	15	231	16	232	14
233	16	234	15	235	15	236	11	237	15	238	14	239	15	240	11
241	14	242	11	243	11	244	5	245	11	246	12	247	16	248	12
249	17	250	16	251	17	252	11	253	15	254	17	255	19	256	15
257	18	258	17	259	17	260	11	261	15	262	18	263	21	264	17
265	20	266	20	267	20	268	14	269	16	270	18	271	19	272	15
273	17	274	15	275	15	276	11	277	16	278	18	279	20	280	18
281	21	282	20	283	18	284	15	285	18	286	20	287	19	288	18
289	19	290	18	291	15	292	15	293	18	294	21	295	18	296	20
297	19	298	20	299	14	300	16	301	16	302	18	303	14	304	16
305	14	306	15	307	11	308	16	309	17	310	20	311	18	312	19
313	19	314	18	315	15	316	17	317	18	318	19	319	18	320	17
321	17	322	16	323	15	324	17	325	18	326	20	327	20	328	17
329	18	330	16	331	16	332	16	333	16	334	17	335	16	336	15
337	15	338	15	339	16	340	18	341	19	342	20	343	19	344	16
345	17	346	16	347	17	348	16	349	18	350	18	351	17	352	14
353	16	354	15	355	17	356	14	357	19	358	17	359	17	360	11
361	15	362	15	363	16	364	11	365	15	366	15	367	16	368	11
369	15	370	18	371	20	372	17	373	19	374	20	375	19	376	14
377	16	378	20	379	20	380	17	381	17	382	20	383	19	384	15
385	15	386	20	387	20	388	19	389	18	390	20	391	18	392	15
393	17	394	19	395	18	396	15	397	17	398	18	399	17	400	15
401	18	402	22	403	20	404	19	405	19	406	20	407	17	408	16
409	19	410	21	411	18	412	17	413	18	414	19	415	15	416	15
417	16	418	20	419	16	420	18	421	16	422	18	423	14	424	16
425	16	426	18	427	14	428	16	429	16	430	17	431	15	432	15
433	18	434	19	435	17	436	17	437	17	438	17	439	16	440	15
441	17	442	17	443	16	444	15	445	16	446	15	447	15	448	11
449	15	450	15	451	16	452	15	453	16	454	16	455	16	456	14
457	16	458	16	459	16	460	15	461	16	462	15	463	15	464	11
465	15	466	15	467	16	468	14	469	16	470	15	471	15	472	11
473	15	474	14	475	15	476	11	477	14	478	11	479	11	480	5
481	11	482	12	483	16	484	12	485	18	486	17	487	19	488	12

Table B.3: *The Abelian complexity $\mathcal{AC}(n)$ of the 5-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
489	18	490	19	491	22	492	16	493	20	494	18	495	18	496	11
497	15	498	17	499	20	500	16	501	20	502	19	503	20	504	14
505	18	506	18	507	20	508	15	509	18	510	15	511	15	512	11
513	16	514	17	515	19	516	18	517	22	518	21	519	19	520	18
521	22	522	23	523	21	524	20	525	21	526	19	527	15	528	15
529	18	530	19	531	18	532	19	533	20	534	19	535	15	536	17
537	18	538	18	539	15	540	16	541	15	542	15	543	11	544	16
545	16	546	19	547	18	548	20	549	20	550	20	551	18	552	21
553	21	554	21	555	20	556	18	557	18	558	16	559	15	560	18
561	17	562	20	563	19	564	19	565	18	566	18	567	17	568	19
569	17	570	18	571	16	572	15	573	15	574	15	575	16	576	18
577	19	578	21	579	20	580	18	581	20	582	20	583	21	584	19
585	21	586	20	587	18	588	14	589	16	590	17	591	18	592	16
593	18	594	19	595	18	596	14	597	16	598	18	599	18	600	14
601	16	602	16	603	16	604	11	605	16	606	20	607	22	608	18
609	20	610	23	611	23	612	18	613	19	614	24	615	23	616	20
617	18	618	20	619	18	620	15	621	18	622	22	623	22	624	18
625	20	626	21	627	21	628	16	629	19	630	21	631	20	632	16
633	17	634	17	635	16	636	16	637	21	638	24	639	21	640	20
641	23	642	24	643	21	644	19	645	22	646	24	647	20	648	18
649	18	650	18	651	16	652	17	653	20	654	21	655	17	656	18
657	19	658	19	659	16	660	16	661	18	662	19	663	15	664	16
665	15	666	17	667	16	668	18	669	20	670	21	671	19	672	20
673	21	674	20	675	19	676	16	677	19	678	18	679	17	680	16
681	16	682	17	683	17	684	16	685	17	686	18	687	17	688	18
689	17	690	17	691	16	692	14	693	16	694	16	695	16	696	15
697	17	698	17	699	18	700	14	701	19	702	19	703	20	704	17
705	19	706	17	707	16	708	11	709	15	710	16	711	18	712	15
713	18	714	17	715	17	716	11	717	15	718	17	719	19	720	15
721	18	722	17	723	17	724	11	725	15	726	18	727	22	728	18
729	22	730	22	731	23	732	17	733	19	734	22	735	23	736	19
737	20	738	19	739	18	740	14	741	17	742	20	743	22	744	19
745	22	746	20	747	20	748	15	749	19	750	20	751	21	752	18
753	20	754	18	755	16	756	15	757	18	758	22	759	20	760	22
761	22	762	23	763	19	764	19	765	21	766	22	767	20	768	19
769	19	770	18	771	15	772	17	773	19	774	21	775	19	776	19
777	19	778	18	779	15	780	17	781	18	782	19	783	18	784	17
785	17	786	16	787	15	788	18	789	19	790	22	791	22	792	20
793	21	794	19	795	19	796	19	797	19	798	20	799	19	800	17
801	17	802	16	803	17	804	19	805	19	806	21	807	19	808	18
809	17	810	17	811	17	812	18	813	18	814	19	815	17	816	15
817	16	818	15	819	18	820	16	821	21	822	20	823	20	824	16
825	18	826	19	827	19	828	16	829	18	830	19	831	18	832	14

Table B.3: *The Abelian complexity* $\mathcal{AC}(n)$ *of the* 5*-bonacci word for* $n \in \{1, \dots, 1000\}$.

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
833	16	834	19	835	20	836	17	837	19	838	20	839	19	840	14
841	16	842	20	843	20	844	17	845	17	846	20	847	19	848	15
849	15	850	21	851	21	852	21	853	19	854	23	855	20	856	18
857	18	858	22	859	20	860	18	861	18	862	20	863	18	864	16
865	18	866	22	867	20	868	19	869	19	870	20	871	17	872	16
873	19	874	21	875	18	876	17	877	18	878	19	879	15	880	15
881	17	882	21	883	18	884	19	885	19	886	20	887	17	888	17
889	19	890	20	891	17	892	17	893	18	894	18	895	16	896	15
897	18	898	19	899	17	900	17	901	17	902	17	903	16	904	15
905	17	906	17	907	16	908	15	909	16	910	15	911	15	912	11
913	15	914	15	915	16	916	15	917	16	918	16	919	16	920	14
921	16	922	16	923	16	924	15	925	16	926	15	927	15	928	11
929	15	930	15	931	16	932	14	933	16	934	15	935	15	936	11
937	15	938	14	939	15	940	11	941	14	942	11	943	11	944	5
945	11	946	11	947	15	948	11	949	17	950	15	951	17	952	11
953	18	954	17	955	21	956	15	957	21	958	17	959	18	960	11
961	18	962	17	963	22	964	16	965	23	966	19	967	21	968	14
969	20	970	18	971	21	972	15	973	19	974	15	975	15	976	11
977	15	978	15	979	18	980	16	981	20	982	18	983	18	984	17
985	20	986	20	987	20	988	19	989	20	990	18	991	15	992	17
993	18	994	19	995	18	996	20	997	20	998	19	999	15	1000	18

Table B.3: *The Abelian complexity $\mathcal{AC}(n)$ of the 5-bonacci word for $n \in \{1, \ldots, 1000\}$.*

B.4 6-bonacci word

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
1	6	2	6	3	10	4	6	5	13	6	10	7	13	8	6
9	15	10	13	11	18	12	10	13	18	14	13	15	15	16	6
17	16	18	15	19	21	20	13	21	23	22	18	23	21	24	10
25	21	26	18	27	23	28	13	29	21	30	15	31	16	32	6
33	16	34	16	35	22	36	15	37	25	38	21	39	24	40	13
41	25	42	23	43	28	44	18	45	27	46	21	47	22	48	10
49	22	50	21	51	27	52	18	53	28	54	23	55	25	56	13
57	24	58	21	59	25	60	15	61	22	62	16	63	16	64	6
65	16	66	16	67	22	68	16	69	25	70	22	71	24	72	16
73	26	74	25	75	28	76	22	77	28	78	24	79	22	80	16
81	25	82	26	83	28	84	25	85	30	86	28	87	25	88	22
89	28	90	28	91	26	92	24	93	25	94	22	95	16	96	16
97	22	98	25	99	24	100	26	101	28	102	28	103	22	104	25
105	28	106	30	107	25	108	28	109	26	110	25	111	16	112	22

Table B.4: *The Abelian complexity $\mathcal{AC}(n)$ of the 6-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
113	24	114	28	115	22	116	28	117	25	118	26	119	16	120	24
121	22	122	25	123	16	124	22	125	16	126	16	127	6	128	16
129	16	130	22	131	16	132	25	133	22	134	25	135	16	136	26
137	25	138	29	139	22	140	28	141	24	142	25	143	16	144	26
145	26	146	30	147	25	148	30	149	28	150	29	151	22	152	29
153	28	154	29	155	24	156	25	157	22	158	22	159	16	160	25
161	25	162	29	163	26	164	29	165	28	166	29	167	25	168	30
169	30	170	30	171	28	172	26	173	25	174	25	175	22	176	28
177	28	178	29	179	28	180	26	181	26	182	25	183	24	184	25
185	25	186	22	187	22	188	16	189	16	190	16	191	16	192	22
193	22	194	25	195	25	196	24	197	25	198	26	199	26	200	28
201	29	202	28	203	28	204	22	205	25	206	25	207	26	208	28
209	30	210	30	211	30	212	25	213	29	214	28	215	29	216	26
217	29	218	25	219	25	220	16	221	22	222	22	223	25	224	24
225	29	226	28	227	29	228	22	229	29	230	28	231	30	232	25
233	30	234	26	235	26	236	16	237	25	238	24	239	28	240	22
241	29	242	25	243	26	244	16	245	25	246	22	247	25	248	16
249	22	250	16	251	16	252	6	253	16	254	17	255	23	256	16
257	25	258	23	259	26	260	16	261	26	262	27	263	31	264	22
265	29	266	26	267	27	268	16	269	26	270	29	271	33	272	25
273	31	274	31	275	32	276	22	277	29	278	31	279	32	280	24
281	28	282	26	283	26	284	16	285	26	286	30	287	34	288	26
289	31	290	33	291	34	292	25	293	30	294	35	295	35	296	28
297	30	298	31	299	31	300	22	301	29	302	34	303	35	304	28
305	29	306	32	307	31	308	24	309	25	310	29	311	26	312	22
313	22	314	22	315	22	316	16	317	26	318	29	319	31	320	25
321	30	322	32	323	32	324	26	325	31	326	36	327	33	328	29
329	30	330	32	331	30	332	26	333	33	334	38	335	35	336	31
337	32	338	36	339	32	340	29	341	29	342	35	343	28	344	28
345	26	346	29	347	25	348	26	349	32	350	37	351	31	352	31
353	32	354	36	355	30	356	30	357	31	358	37	359	28	360	30
361	27	362	32	363	25	364	29	365	30	366	36	367	26	368	29
369	25	370	31	371	22	372	25	373	20	374	26	375	16	376	22
377	16	378	22	379	16	380	25	381	25	382	31	383	22	384	29
385	27	386	32	387	25	388	29	389	29	390	34	391	25	392	30
393	27	394	31	395	26	396	30	397	33	398	36	399	29	400	31
401	31	402	33	403	28	404	26	405	28	406	31	407	25	408	28
409	25	410	29	411	26	412	29	413	32	414	34	415	30	416	31
417	32	418	32	419	30	420	26	421	31	422	32	423	29	424	29
425	29	426	30	427	29	428	26	429	31	430	30	431	29	432	25
433	28	434	25	435	25	436	16	437	22	438	22	439	22	440	22
441	22	442	25	443	25	444	24	445	28	446	29	447	29	448	28
449	30	450	29	451	29	452	22	453	29	454	29	455	30	456	28

Table B.4: *The Abelian complexity $\mathcal{AC}(n)$ of the 6-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
457	30	458	30	459	30	460	25	461	31	462	30	463	31	464	26
465	30	466	26	467	26	468	16	469	25	470	25	471	28	472	24
473	29	474	28	475	29	476	22	477	30	478	29	479	31	480	25
481	30	482	26	483	26	484	16	485	26	486	25	487	29	488	22
489	29	490	25	491	26	492	16	493	25	494	22	495	25	496	16
497	22	498	16	499	16	500	6	501	16	502	17	503	24	504	17
505	27	506	24	507	27	508	16	509	26	510	27	511	32	512	23
513	31	514	27	515	28	516	16	517	26	518	29	519	35	520	27
521	35	522	33	523	34	524	22	525	30	526	32	527	35	528	26
529	32	530	28	531	28	532	16	533	26	534	30	535	37	536	29
537	37	538	36	539	37	540	25	541	31	542	36	543	39	544	31
545	36	546	34	547	34	548	22	549	29	550	34	551	38	552	31
553	35	554	35	555	34	556	24	557	28	558	31	559	32	560	25
561	29	562	25	563	25	564	16	565	27	566	30	567	36	568	30
569	38	570	36	571	35	572	26	573	33	574	37	575	38	576	33
577	37	578	35	579	32	580	26	581	33	582	39	583	39	584	37
585	39	586	40	587	34	588	30	589	33	590	38	591	34	592	33
593	33	594	32	595	26	596	26	597	33	598	39	599	36	600	38
601	39	602	40	603	31	604	31	605	34	606	40	607	33	608	36
609	33	610	35	611	25	612	29	613	30	614	38	615	29	616	35
617	29	618	34	619	22	620	28	621	25	622	31	623	22	624	28
625	22	626	25	627	16	628	27	629	28	630	36	631	29	632	36
633	33	634	35	635	25	636	32	637	33	638	38	639	32	640	35
641	33	642	33	643	26	644	32	645	35	646	41	647	36	648	37
649	36	650	36	651	29	652	32	653	33	654	36	655	32	656	32
657	30	658	30	659	26	660	33	661	35	662	40	663	38	664	36
665	37	666	34	667	31	668	32	669	35	670	37	671	36	672	32
673	33	674	30	675	29	676	29	677	33	678	35	679	35	680	28
681	31	682	28	683	28	684	26	685	28	686	29	687	28	688	25
689	25	690	26	691	27	692	32	693	34	694	37	695	36	696	31
697	33	698	31	699	32	700	32	701	35	702	36	703	35	704	30
705	32	706	30	707	32	708	31	709	37	710	37	711	37	712	28
713	33	714	30	715	32	716	27	717	33	718	32	719	32	720	25
721	30	722	29	723	33	724	30	725	38	726	36	727	36	728	26
729	32	730	29	731	32	732	25	733	34	734	31	735	32	736	22
737	29	738	25	739	29	740	20	741	31	742	26	743	28	744	16
745	25	746	23	747	26	748	16	749	25	750	23	751	26	752	16
753	26	754	28	755	34	756	26	757	35	758	33	759	34	760	22
761	30	762	33	763	36	764	28	765	34	766	35	767	35	768	25
769	30	770	34	771	37	772	31	773	37	774	37	775	36	776	25
777	31	778	35	779	36	780	28	781	32	782	35	783	35	784	26
785	30	786	38	787	40	788	36	789	37	790	41	791	38	792	29
793	31	794	38	795	38	796	33	797	33	798	38	799	35	800	28

Table B.4: *The Abelian complexity $\mathcal{AC}(n)$ of the 6-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
801	26	802	33	803	32	804	31	805	31	806	34	807	31	808	25
809	29	810	32	811	31	812	25	813	30	814	32	815	32	816	26
817	32	818	39	819	38	820	35	821	36	822	39	823	35	824	30
825	35	826	41	827	38	828	34	829	35	830	39	831	34	832	30
833	31	834	40	835	36	836	37	837	35	838	39	839	32	840	31
841	34	842	39	843	33	844	32	845	34	846	37	847	31	848	30
849	34	850	42	851	36	852	37	853	35	854	39	855	30	856	31
857	32	858	39	859	31	860	33	861	30	862	35	863	25	864	26
865	23	866	32	867	25	868	31	869	25	870	31	871	22	872	28
873	25	874	31	875	22	876	29	877	27	878	32	879	25	880	29
881	31	882	37	883	30	884	34	885	32	886	35	887	29	888	31
889	34	890	37	891	31	892	32	893	33	894	34	895	29	896	26
897	31	898	35	899	32	900	33	901	32	902	34	903	30	904	30
905	32	906	34	907	30	908	31	909	32	910	32	911	30	912	26
913	33	914	34	915	33	916	31	917	33	918	32	919	31	920	26
921	32	922	31	923	31	924	26	925	30	926	26	927	26	928	16
929	25	930	25	931	28	932	25	933	28	934	28	935	28	936	24
937	28	938	29	939	29	940	28	941	30	942	29	943	29	944	22
945	30	946	30	947	32	948	29	949	32	950	31	951	31	952	25
953	31	954	30	955	31	956	26	957	30	958	26	959	26	960	16
961	26	962	26	963	30	964	25	965	31	966	29	967	30	968	22
969	30	970	29	971	31	972	25	973	30	974	26	975	26	976	16
977	26	978	25	979	29	980	22	981	29	982	25	983	26	984	16
985	25	986	22	987	25	988	16	989	22	990	16	991	16	992	6
993	16	994	17	995	24	996	17	997	28	998	25	999	29	1000	17

Table B.4: *The Abelian complexity $\mathcal{AC}(n)$ of the 6-bonacci word for $n \in \{1, \ldots, 1000\}$.*

B.5 7-bonacci word

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
1	7	2	7	3	12	4	7	5	16	6	12	7	16	8	7
9	19	10	16	11	23	12	12	13	23	14	16	15	19	16	7
17	21	18	19	19	28	20	16	21	31	22	23	23	28	24	12
25	28	26	23	27	31	28	16	29	28	30	19	31	21	32	7
33	22	34	21	35	31	36	19	37	36	38	28	39	34	40	16
41	36	42	31	43	41	44	23	45	39	46	28	47	31	48	12
49	31	50	28	51	39	52	23	53	41	54	31	55	36	56	16
57	34	58	28	59	36	60	19	61	31	62	21	63	22	64	7
65	22	66	22	67	32	68	21	69	38	70	31	71	37	72	19
73	40	74	36	75	46	76	28	77	45	78	34	79	37	80	16

Table B.5: *The Abelian complexity $\mathcal{AC}(n)$ of the 7-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
81	38	82	36	83	48	84	31	85	51	86	41	87	46	88	23
89	45	90	39	91	48	92	28	93	43	94	31	95	32	96	12
97	32	98	31	99	43	100	28	101	48	102	39	103	45	104	23
105	46	106	41	107	51	108	31	109	48	110	36	111	38	112	16
113	37	114	34	115	45	116	28	117	46	118	36	119	40	120	19
121	37	122	31	123	38	124	21	125	32	126	22	127	22	128	7
129	22	130	22	131	32	132	22	133	38	134	32	135	37	136	22
137	41	138	38	139	46	140	32	141	46	142	37	143	37	144	22
145	41	146	41	147	49	148	38	149	53	150	46	151	46	152	32
153	49	154	46	155	49	156	37	157	46	158	37	159	32	160	22
161	38	162	41	163	46	164	41	165	53	166	49	167	46	168	38
169	53	170	53	171	53	172	46	173	53	174	46	175	38	176	32
177	46	178	49	179	49	180	46	181	53	182	49	183	41	184	37
185	46	186	46	187	41	188	37	189	38	190	32	191	22	192	22
193	32	194	38	195	37	196	41	197	46	198	46	199	37	200	41
201	49	202	53	203	46	204	49	205	49	206	46	207	32	208	38
209	46	210	53	211	46	212	53	213	53	214	53	215	38	216	46
217	49	218	53	219	41	220	46	221	41	222	38	223	22	224	32
225	37	226	46	227	37	228	49	229	46	230	49	231	32	232	46
233	46	234	53	235	38	236	49	237	41	238	41	239	22	240	37
241	37	242	46	243	32	244	46	245	38	246	41	247	22	248	37
249	32	250	38	251	22	252	32	253	22	254	22	255	7	256	22
257	22	258	32	259	22	260	38	261	32	262	38	263	22	264	41
265	38	266	47	267	32	268	46	269	37	270	40	271	22	272	42
273	41	274	51	275	38	276	53	277	46	278	50	279	32	280	50
281	46	282	52	283	37	284	46	285	37	286	38	287	22	288	41
289	41	290	51	291	41	292	54	293	49	294	53	295	38	296	55
297	53	298	58	299	46	300	53	301	46	302	47	303	32	304	50
305	49	306	56	307	46	308	54	309	49	310	50	311	37	312	49
313	46	314	47	315	37	316	38	317	32	318	32	319	22	320	38
321	38	322	47	323	41	324	49	325	46	326	50	327	41	328	54
329	53	330	56	331	49	332	50	333	46	334	47	335	38	336	53
337	53	338	58	339	53	340	55	341	53	342	53	343	46	344	54
345	53	346	51	347	46	348	41	349	38	350	38	351	32	352	46
353	46	354	52	355	49	356	50	357	49	358	50	359	46	360	53
361	53	362	51	363	49	364	42	365	41	366	40	367	37	368	46
369	46	370	47	371	46	372	41	373	41	374	38	375	37	376	38
377	38	378	32	379	32	380	22	381	22	382	22	383	22	384	32
385	32	386	38	387	38	388	37	389	38	390	41	391	41	392	46
393	47	394	46	395	46	396	37	397	40	398	41	399	42	400	49
401	51	402	53	403	53	404	46	405	50	406	49	407	50	408	49
409	52	410	46	411	46	412	32	413	38	414	38	415	41	416	46
417	51	418	53	419	54	420	46	421	53	422	53	423	55	424	53

Table B.5: *The Abelian complexity $\mathcal{AC}(n)$ of the 7-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
425	58	426	53	427	53	428	38	429	47	430	46	431	50	432	49
433	56	434	53	435	54	436	41	437	50	438	46	439	49	440	41
441	47	442	38	443	38	444	22	445	32	446	32	447	38	448	37
449	47	450	46	451	49	452	37	453	50	454	49	455	54	456	46
457	56	458	49	459	50	460	32	461	47	462	46	463	53	464	46
465	58	466	53	467	55	468	38	469	53	470	49	471	54	472	41
473	51	474	41	475	41	476	22	477	38	478	37	479	46	480	37
481	52	482	46	483	50	484	32	485	50	486	46	487	53	488	38
489	51	490	41	491	42	492	22	493	40	494	37	495	46	496	32
497	47	498	38	499	41	500	22	501	38	502	32	503	38	504	22
505	32	506	22	507	22	508	7	509	22	510	23	511	33	512	22
513	38	514	33	515	39	516	22	517	41	518	40	519	49	520	32
521	47	522	39	523	42	524	22	525	42	526	44	527	54	528	38
529	54	530	49	531	53	532	32	533	50	534	49	535	55	536	37
537	49	538	41	539	42	540	22	541	42	542	46	543	56	544	41
545	56	546	54	547	58	548	38	549	55	550	58	551	63	552	46
553	57	554	52	555	53	556	32	557	51	558	55	559	62	560	46
561	57	562	55	563	56	564	37	565	49	566	50	567	51	568	37
569	44	570	39	571	39	572	22	573	41	574	46	575	55	576	41
577	54	578	55	579	59	580	41	581	55	582	61	583	64	584	49
585	57	586	56	587	57	588	38	589	55	590	63	591	68	592	53
593	60	594	63	595	63	596	46	597	54	598	60	599	58	600	46
601	50	602	49	603	49	604	32	605	50	606	57	607	63	608	49
609	57	610	61	611	62	612	46	613	54	614	62	615	60	616	49
617	51	618	53	619	52	620	37	621	49	622	56	623	57	624	46
625	47	626	51	627	48	628	37	629	38	630	43	631	37	632	32
633	32	634	32	635	32	636	22	637	39	638	43	639	48	640	38
641	48	642	51	643	53	644	41	645	51	646	58	647	55	648	47
649	51	650	54	651	53	652	42	653	57	654	65	655	64	656	54
657	58	658	64	659	60	660	50	661	53	662	62	663	53	664	49
665	48	666	52	667	48	668	42	669	57	670	66	671	63	672	56
673	61	674	68	675	63	676	55	677	60	678	71	679	61	680	57
681	55	682	62	683	55	684	51	685	59	686	70	687	61	688	57
689	54	690	63	691	53	692	49	693	45	694	55	695	42	696	44
697	39	698	45	699	38	700	41	701	51	702	61	703	52	704	54
705	56	706	65	707	55	708	55	709	57	710	69	711	54	712	57
713	53	714	62	715	51	716	55	717	61	718	73	719	58	720	60
721	56	722	67	723	53	724	54	725	49	726	61	727	44	728	50
729	42	730	52	731	40	732	50	733	53	734	66	735	49	736	57
737	52	738	64	739	48	740	54	741	49	742	62	743	43	744	51
745	41	746	53	747	38	748	49	749	45	750	58	751	39	752	47
753	36	754	48	755	32	756	38	757	27	758	37	759	22	760	32
761	22	762	32	763	22	764	38	765	36	766	48	767	32	768	47

Table B.5: *The Abelian complexity $\mathcal{AC}(n)$ of the 7-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
769	41	770	53	771	38	772	49	773	44	774	56	775	38	776	51
777	43	778	54	779	41	780	54	781	54	782	65	783	47	784	57
785	51	786	61	787	46	788	50	789	46	790	56	791	40	792	50
793	42	794	52	795	42	796	54	797	57	798	66	799	51	800	60
801	58	802	65	803	53	804	55	805	57	806	65	807	50	808	57
809	52	810	60	811	50	812	55	813	59	814	65	815	52	816	54
817	52	818	56	819	46	820	41	821	42	822	48	823	38	824	44
825	38	826	47	827	41	828	49	829	53	830	60	831	51	832	57
833	57	834	61	835	54	836	51	837	57	838	62	839	53	840	57
841	55	842	60	843	55	844	55	845	63	846	65	847	58	848	56
849	59	850	58	851	53	852	42	853	50	854	53	855	47	856	49
857	47	858	53	859	50	860	50	861	58	862	60	863	56	864	54
865	58	866	56	867	54	868	42	869	53	870	53	871	50	872	47
873	49	874	50	875	49	876	41	877	51	878	48	879	47	880	38
881	44	882	38	883	38	884	22	885	32	886	32	887	32	888	32
889	32	890	38	891	38	892	37	893	44	894	47	895	47	896	46
897	50	898	49	899	49	900	37	901	49	902	50	903	51	904	49
905	52	906	54	907	54	908	46	909	57	910	56	911	57	912	49
913	56	914	50	915	50	916	32	917	47	918	47	919	50	920	46
921	51	922	53	923	54	924	46	925	58	926	58	927	60	928	53
929	60	930	55	931	55	932	38	933	54	934	53	935	57	936	49
937	57	938	54	939	55	940	41	941	55	942	51	943	54	944	41
945	50	946	41	947	41	948	22	949	38	950	38	951	44	952	37
953	47	954	46	955	49	956	37	957	53	958	52	959	57	960	46
961	57	962	50	963	51	964	32	965	51	966	50	967	57	968	46
969	58	970	53	971	55	972	38	973	55	974	51	975	56	976	41
977	52	978	42	979	42	980	22	981	41	982	40	983	49	984	37
985	52	986	46	987	50	988	32	989	51	990	47	991	54	992	38
993	51	994	41	995	42	996	22	997	41	998	38	999	47	1000	32

Table B.5: *The Abelian complexity $\mathcal{AC}(n)$ of the 7-bonacci word for $n \in \{1, \ldots, 1000\}$.*

B.6 8-bonacci word

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
1	8	2	8	3	14	4	8	5	19	6	14	7	19	8	8
9	23	10	19	11	28	12	14	13	28	14	19	15	23	16	8
17	26	18	23	19	35	20	19	21	39	22	28	23	35	24	14
25	35	26	28	27	39	28	19	29	35	30	23	31	26	32	8
33	28	34	26	35	40	36	23	37	47	38	35	39	44	40	19
41	47	42	39	43	54	44	28	45	51	46	35	47	40	48	14

Table B.6: *The Abelian complexity $\mathcal{AC}(n)$ of the 8-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
49	40	50	35	51	51	52	28	53	54	54	39	55	47	56	19
57	44	58	35	59	47	60	23	61	40	62	26	63	28	64	8
65	29	66	28	67	43	68	26	69	52	70	40	71	50	72	23
73	55	74	47	75	64	76	35	77	62	78	44	79	50	80	19
81	52	82	47	83	67	84	39	85	72	86	54	87	64	88	28
89	62	90	51	91	67	92	35	93	59	94	40	95	43	96	14
97	43	98	40	99	59	100	35	101	67	102	51	103	62	104	28
105	64	106	54	107	72	108	39	109	67	110	47	111	52	112	19
113	50	114	44	115	62	116	35	117	64	118	47	119	55	120	23
121	50	122	40	123	52	124	26	125	43	126	28	127	29	128	8
129	29	130	29	131	44	132	28	133	54	134	43	135	53	136	26
137	59	138	52	139	69	140	40	141	68	142	50	143	56	144	23
145	59	146	55	147	76	148	47	149	82	150	64	151	74	152	35
153	73	154	62	155	79	156	44	157	71	158	50	159	53	160	19
161	54	162	52	163	74	164	47	165	84	166	67	167	79	168	39
169	82	170	72	171	92	172	54	173	87	174	64	175	69	176	28
177	68	178	62	179	84	180	51	181	87	182	67	183	76	184	35
185	71	186	59	187	74	188	40	189	63	190	43	191	44	192	14
193	44	194	43	195	63	196	40	197	74	198	59	199	71	200	35
201	76	202	67	203	87	204	51	205	84	206	62	207	68	208	28
209	69	210	64	211	87	212	54	213	92	214	72	215	82	216	39
217	79	218	67	219	84	220	47	221	74	222	52	223	54	224	19
225	53	226	50	227	71	228	44	229	79	230	62	231	73	232	35
233	74	234	64	235	82	236	47	237	76	238	55	239	59	240	23
241	56	242	50	243	68	244	40	245	69	246	52	247	59	248	26
249	53	250	43	251	54	252	28	253	44	254	29	255	29	256	8
257	29	258	29	259	44	260	29	261	54	262	44	263	53	264	29
265	60	266	54	267	69	268	44	269	69	270	53	271	56	272	29
273	62	274	60	275	77	276	54	277	84	278	69	279	74	280	44
281	77	282	69	283	80	284	53	285	74	286	56	287	53	288	29
289	60	290	62	291	77	292	60	293	89	294	77	295	80	296	54
297	89	298	84	299	94	300	69	301	92	302	74	303	69	304	44
305	77	306	77	307	88	308	69	309	94	310	80	311	77	312	53
313	80	314	74	315	77	316	56	317	69	318	53	319	44	320	29
321	54	322	60	323	69	324	62	325	84	326	77	327	74	328	60
329	89	330	89	331	92	332	77	333	94	334	80	335	69	336	54
337	84	338	89	339	94	340	84	341	104	342	94	343	84	344	69
345	94	346	92	347	89	348	74	349	84	350	69	351	54	352	44
353	69	354	77	355	80	356	77	357	94	358	88	359	77	360	69
361	92	362	94	363	89	364	80	365	89	366	77	367	60	368	53
369	74	370	80	371	77	372	74	373	84	374	77	375	62	376	56
377	69	378	69	379	60	380	53	381	54	382	44	383	29	384	29
385	44	386	54	387	53	388	60	389	69	390	69	391	56	392	62

Table B.6: *The Abelian complexity $\mathcal{AC}(n)$ of the 8-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
393	77	394	84	395	74	396	77	397	80	398	74	399	53	400	60
401	77	402	89	403	80	404	89	405	94	406	92	407	69	408	77
409	88	410	94	411	77	412	80	413	77	414	69	415	44	416	54
417	69	418	84	419	74	420	89	421	92	422	94	423	69	424	84
425	94	426	104	427	84	428	94	429	89	430	84	431	54	432	69
433	80	434	94	435	77	436	92	437	89	438	89	439	60	440	74
441	77	442	84	443	62	444	69	445	60	446	54	447	29	448	44
449	53	450	69	451	56	452	77	453	74	454	80	455	53	456	77
457	80	458	94	459	69	460	88	461	77	462	77	463	44	464	69
465	74	466	92	467	69	468	94	469	84	470	89	471	54	472	80
473	77	474	89	475	60	476	77	477	62	478	60	479	29	480	53
481	56	482	74	483	53	484	80	485	69	486	77	487	44	488	74
489	69	490	84	491	54	492	77	493	60	494	62	495	29	496	56
497	53	498	69	499	44	500	69	501	54	502	60	503	29	504	53
505	44	506	54	507	29	508	44	509	29	510	29	511	8	512	29
513	29	514	44	515	29	516	54	517	44	518	54	519	29	520	60
521	54	522	70	523	44	524	69	525	53	526	59	527	29	528	63
529	60	530	79	531	54	532	84	533	69	534	78	535	44	536	78
537	69	538	83	539	53	540	74	541	56	542	59	543	29	544	63
545	62	546	82	547	60	548	90	549	77	550	87	551	54	552	91
553	84	554	99	555	69	556	92	557	74	558	78	559	44	560	81
561	77	562	95	563	69	564	95	565	80	566	86	567	53	568	83
569	74	570	83	571	56	572	69	573	53	574	54	575	29	576	60
577	60	578	79	579	62	580	87	581	77	582	87	583	60	584	94
585	89	586	102	587	77	588	95	589	80	590	84	591	54	592	91
593	89	594	106	595	84	596	106	597	94	598	99	599	69	600	99
601	92	602	99	603	74	604	84	605	69	606	70	607	44	608	78
609	77	610	95	611	77	612	98	613	88	614	95	615	69	616	99
617	94	618	102	619	80	620	90	621	77	622	78	623	53	624	83
625	80	626	92	627	74	628	87	629	77	630	78	631	56	632	75
633	69	634	70	635	53	636	54	637	44	638	44	639	29	640	54
641	54	642	70	643	60	644	75	645	69	646	78	647	62	648	87
649	84	650	92	651	77	652	83	653	74	654	78	655	60	656	90
657	89	658	102	659	89	660	99	661	92	662	95	663	77	664	98
665	94	666	95	667	80	668	78	669	69	670	70	671	54	672	84
673	84	674	99	675	89	676	99	677	94	678	99	679	84	680	106
681	104	682	106	683	94	684	91	685	84	686	84	687	69	688	95
689	94	690	102	691	92	692	94	693	89	694	87	695	74	696	87
697	84	698	79	699	69	700	60	701	54	702	54	703	44	704	69
705	69	706	83	707	77	708	83	709	80	710	86	711	77	712	95
713	94	714	95	715	88	716	81	717	77	718	78	719	69	720	92
721	92	722	99	723	94	724	91	725	89	726	87	727	80	728	90
729	89	730	82	731	77	732	63	733	60	734	59	735	53	736	74

Table B.6: *The Abelian complexity $\mathcal{AC}(n)$ of the 8-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
737	74	738	83	739	80	740	78	741	77	742	78	743	74	744	84
745	84	746	79	747	77	748	63	749	62	750	59	751	56	752	69
753	69	754	70	755	69	756	60	757	60	758	54	759	53	760	54
761	54	762	44	763	44	764	29	765	29	766	29	767	29	768	44
769	44	770	54	771	54	772	53	773	54	774	60	775	60	776	69
777	70	778	69	779	69	780	56	781	59	782	62	783	63	784	77
785	79	786	84	787	84	788	74	789	78	790	77	791	78	792	80
793	83	794	74	795	74	796	53	797	59	798	60	799	63	800	77
801	82	802	89	803	90	804	80	805	87	806	89	807	91	808	94
809	99	810	92	811	92	812	69	813	78	814	77	815	81	816	88
817	95	818	94	819	95	820	77	821	86	822	80	823	83	824	77
825	83	826	69	827	69	828	44	829	54	830	54	831	60	832	69
833	79	834	84	835	87	836	74	837	87	838	89	839	94	840	92
841	102	842	94	843	95	844	69	845	84	846	84	847	91	848	94
849	106	850	104	851	106	852	84	853	99	854	94	855	99	856	89
857	99	858	84	859	84	860	54	861	70	862	69	863	78	864	80
865	95	866	94	867	98	868	77	869	95	870	92	871	99	872	89
873	102	874	89	875	90	876	60	877	78	878	74	879	83	880	77
881	92	882	84	883	87	884	62	885	78	886	69	887	75	888	60
889	70	890	54	891	54	892	29	893	44	894	44	895	54	896	53
897	70	898	69	899	75	900	56	901	78	902	77	903	87	904	74
905	92	906	80	907	83	908	53	909	78	910	77	911	90	912	80
913	102	914	94	915	99	916	69	917	95	918	88	919	98	920	77
921	95	922	77	923	78	924	44	925	70	926	69	927	84	928	74
929	99	930	92	931	99	932	69	933	99	934	94	935	106	936	84
937	106	938	89	939	91	940	54	941	84	942	80	943	95	944	77
945	102	946	89	947	94	948	60	949	87	950	77	951	87	952	62
953	79	954	60	955	60	956	29	957	54	958	53	959	69	960	56
961	83	962	74	963	83	964	53	965	86	966	80	967	95	968	69
969	95	970	77	971	81	972	44	973	78	974	74	975	92	976	69
977	99	978	84	979	91	980	54	981	87	982	77	983	90	984	60
985	82	986	62	987	63	988	29	989	59	990	56	991	74	992	53
993	83	994	69	995	78	996	44	997	78	998	69	999	84	1000	54

Table B.6: *The Abelian complexity $\mathcal{AC}(n)$ of the 8-bonacci word for $n \in \{1, \dots, 1000\}$.*

B.7 9-bonacci word

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
1	9	2	9	3	16	4	9	5	22	6	16	7	22	8	9
9	27	10	22	11	33	12	16	13	33	14	22	15	27	16	9

Table B.7: *The Abelian complexity $\mathcal{AC}(n)$ of the 9-bonacci word for $n \in \{1, \dots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
17	31	18	27	19	42	20	22	21	47	22	33	23	42	24	16
25	42	26	33	27	47	28	22	29	42	30	27	31	31	32	9
33	34	34	31	35	49	36	27	37	58	38	42	39	54	40	22
41	58	42	47	43	67	44	33	45	63	46	42	47	49	48	16
49	49	50	42	51	63	52	33	53	67	54	47	55	58	56	22
57	54	58	42	59	58	60	27	61	49	62	31	63	34	64	9
65	36	66	34	67	54	68	31	69	66	70	49	71	63	72	27
73	70	74	58	75	82	76	42	77	79	78	54	79	63	80	22
81	66	82	58	83	86	84	47	85	93	86	67	87	82	88	33
89	79	90	63	91	86	92	42	93	75	94	49	95	54	96	16
97	54	98	49	99	75	100	42	101	86	102	63	103	79	104	33
105	82	106	67	107	93	108	47	109	86	110	58	111	66	112	22
113	63	114	54	115	79	116	42	117	82	118	58	119	70	120	27
121	63	122	49	123	66	124	31	125	54	126	34	127	36	128	9
129	37	130	36	131	57	132	34	133	71	134	54	135	69	136	31
137	78	138	66	139	92	140	49	141	90	142	63	143	73	144	27
145	78	146	70	147	102	148	58	149	111	150	82	151	99	152	42
153	97	154	79	155	106	156	54	157	94	158	63	159	69	160	22
161	71	162	66	163	99	164	58	165	114	166	86	167	106	168	47
169	111	170	93	171	126	172	67	173	118	174	82	175	92	176	33
177	90	178	79	179	113	180	63	181	118	182	86	183	102	184	42
185	94	186	75	187	99	188	49	189	83	190	54	191	57	192	16
193	57	194	54	195	83	196	49	197	99	198	75	199	94	200	42
201	102	202	86	203	118	204	63	205	113	206	79	207	90	208	33
209	92	210	82	211	118	212	67	213	126	214	93	215	111	216	47
217	106	218	86	219	114	220	58	221	99	222	66	223	71	224	22
225	69	226	63	227	94	228	54	229	106	230	79	231	97	232	42
233	99	234	82	235	111	236	58	237	102	238	70	239	78	240	27
241	73	242	63	243	90	244	49	245	92	246	66	247	78	248	31
249	69	250	54	251	71	252	34	253	57	254	36	255	37	256	9
257	37	258	37	259	58	260	36	261	73	262	57	263	72	264	34
265	82	266	71	267	97	268	54	269	96	270	69	271	79	272	31
273	85	274	78	275	111	276	66	277	121	278	92	279	109	280	49
281	108	282	90	283	118	284	63	285	106	286	73	287	79	288	27
289	82	290	78	291	114	292	70	293	131	294	102	295	123	296	58
297	129	298	111	299	146	300	82	301	138	302	99	303	109	304	42
305	108	306	97	307	135	308	79	309	141	310	106	311	123	312	54
313	115	314	94	315	121	316	63	317	103	318	69	319	72	320	22
321	73	322	71	323	106	324	66	325	126	326	99	327	121	328	58
329	131	330	114	331	151	332	86	333	146	334	106	335	118	336	47
337	121	338	111	339	154	340	93	341	164	342	126	343	146	344	67
345	141	346	118	347	151	348	82	349	133	350	92	351	97	352	33
353	96	354	90	355	130	356	79	357	146	358	113	359	135	360	63

Table B.7: *The Abelian complexity $\mathcal{AC}(n)$ of the 9-bonacci word for $n \in \{1, \dots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
361	138	362	118	363	154	364	86	365	143	366	102	367	111	368	42
369	106	370	94	371	130	372	75	373	133	374	99	375	114	376	49
377	103	378	83	379	106	380	54	381	87	382	57	383	58	384	16
385	58	386	57	387	87	388	54	389	106	390	83	391	103	392	49
393	114	394	99	395	133	396	75	397	130	398	94	399	106	400	42
401	111	402	102	403	143	404	86	405	154	406	118	407	138	408	63
409	135	410	113	411	146	412	79	413	130	414	90	415	96	416	33
417	97	418	92	419	133	420	82	421	151	422	118	423	141	424	67
425	146	426	126	427	164	428	93	429	154	430	111	431	121	432	47
433	118	434	106	435	146	436	86	437	151	438	114	439	131	440	58
441	121	442	99	443	126	444	66	445	106	446	71	447	73	448	22
449	72	450	69	451	103	452	63	453	121	454	94	455	115	456	54
457	123	458	106	459	141	460	79	461	135	462	97	463	108	464	42
465	109	466	99	467	138	468	82	469	146	470	111	471	129	472	58
473	123	474	102	475	131	476	70	477	114	478	78	479	82	480	27
481	79	482	73	483	106	484	63	485	118	486	90	487	108	488	49
489	109	490	92	491	121	492	66	493	111	494	78	495	85	496	31
497	79	498	69	499	96	500	54	501	97	502	71	503	82	504	34
505	72	506	57	507	73	508	36	509	58	510	37	511	37	512	9
513	37	514	37	515	58	516	37	517	73	518	58	519	72	520	37
521	83	522	73	523	97	524	58	525	97	526	72	527	79	528	37
529	88	530	83	531	112	532	73	533	123	534	97	535	109	536	58
537	112	538	97	539	119	540	72	541	109	542	79	543	79	544	37
545	88	546	88	547	117	548	83	549	136	550	112	551	124	552	73
553	136	554	123	555	148	556	97	557	143	558	109	559	109	560	58
561	117	562	112	563	139	564	97	565	148	566	119	567	124	568	72
569	124	570	109	571	124	572	79	573	109	574	79	575	72	576	37
577	83	578	88	579	112	580	88	581	136	582	117	583	124	584	83
585	144	586	136	587	156	588	112	589	156	590	124	591	119	592	73
593	136	594	136	595	161	596	123	597	176	598	148	599	148	600	97
601	156	602	143	603	156	604	109	605	143	606	109	607	97	608	58
609	112	610	117	611	139	612	112	613	161	614	139	615	139	616	97
617	156	618	148	619	161	620	119	621	156	622	124	623	112	624	72
625	124	626	124	627	139	628	109	629	148	630	124	631	117	632	79
633	119	634	109	635	112	636	79	637	97	638	72	639	58	640	37
641	73	642	83	643	97	644	88	645	123	646	112	647	109	648	88
649	136	650	136	651	143	652	117	653	148	654	124	655	109	656	83
657	136	658	144	659	156	660	136	661	176	662	156	663	143	664	112
665	161	666	156	667	156	668	124	669	148	670	119	671	97	672	73
673	123	674	136	675	148	676	136	677	176	678	161	679	148	680	123
681	176	682	176	683	176	684	148	685	176	686	148	687	123	688	97
689	148	690	156	691	161	692	143	693	176	694	156	695	136	696	109
697	148	698	143	699	136	700	109	701	123	702	97	703	73	704	58

Table B.7: *The Abelian complexity $\mathcal{AC}(n)$ of the 9-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
705	97	706	112	707	119	708	117	709	148	710	139	711	124	712	112
713	156	714	161	715	156	716	139	717	161	718	139	719	112	720	97
721	143	722	156	723	156	724	148	725	176	726	161	727	136	728	119
729	156	730	156	731	144	732	124	733	136	734	112	735	83	736	72
737	109	738	124	739	124	740	124	741	148	742	139	743	117	744	109
745	143	746	148	747	136	748	124	749	136	750	117	751	88	752	79
753	109	754	119	755	112	756	109	757	123	758	112	759	88	760	79
761	97	762	97	763	83	764	72	765	73	766	58	767	37	768	37
769	58	770	73	771	72	772	83	773	97	774	97	775	79	776	88
777	112	778	123	779	109	780	112	781	119	782	109	783	79	784	88
785	117	786	136	787	124	788	136	789	148	790	143	791	109	792	117
793	139	794	148	795	124	796	124	797	124	798	109	799	72	800	83
801	112	802	136	803	124	804	144	805	156	806	156	807	119	808	136
809	161	810	176	811	148	812	156	813	156	814	143	815	97	816	112
817	139	818	161	819	139	820	156	821	161	822	156	823	112	824	124
825	139	826	148	827	117	828	119	829	112	830	97	831	58	832	73
833	97	834	123	835	109	836	136	837	143	838	148	839	109	840	136
841	156	842	176	843	143	844	161	845	156	846	148	847	97	848	123
849	148	850	176	851	148	852	176	853	176	854	176	855	123	856	148
857	161	858	176	859	136	860	148	861	136	862	123	863	73	864	97
865	119	866	148	867	124	868	156	869	156	870	161	871	112	872	143
873	156	874	176	875	136	876	156	877	144	878	136	879	83	880	109
881	124	882	148	883	117	884	143	885	136	886	136	887	88	888	109
889	112	890	123	891	88	892	97	893	83	894	73	895	37	896	58
897	72	898	97	899	79	900	112	901	109	902	119	903	79	904	117
905	124	906	148	907	109	908	139	909	124	910	124	911	72	912	112
913	124	914	156	915	119	916	161	917	148	918	156	919	97	920	139
921	139	922	161	923	112	924	139	925	117	926	112	927	58	928	97
929	109	930	143	931	109	932	156	933	143	934	156	935	97	936	148
937	148	938	176	939	123	940	161	941	136	942	136	943	73	944	119
945	124	946	156	947	112	948	156	949	136	950	144	951	83	952	124
953	117	954	136	955	88	956	112	957	88	958	83	959	37	960	72
961	79	962	109	963	79	964	124	965	109	966	124	967	72	968	124
969	119	970	148	971	97	972	139	973	112	974	117	975	58	976	109
977	109	978	143	979	97	980	148	981	123	982	136	983	73	984	124
985	112	986	136	987	83	988	117	989	88	990	88	991	37	992	79
993	79	994	109	995	72	996	119	997	97	998	112	999	58	1000	109

Table B.7: *The Abelian complexity $\mathcal{AC}(n)$ of the 9-bonacci word for $n \in \{1, \ldots, 1000\}$.*

B.8 10-bonacci word

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
1	10	2	10	3	18	4	10	5	25	6	18	7	25	8	10
9	31	10	25	11	38	12	18	13	38	14	25	15	31	16	10
17	36	18	31	19	49	20	25	21	55	22	38	23	49	24	18
25	49	26	38	27	55	28	25	29	49	30	31	31	36	32	10
33	40	34	36	35	58	36	31	37	69	38	49	39	64	40	25
41	69	42	55	43	80	44	38	45	75	46	49	47	58	48	18
49	58	50	49	51	75	52	38	53	80	54	55	55	69	56	25
57	64	58	49	59	69	60	31	61	58	62	36	63	40	64	10
65	43	66	40	67	65	68	36	69	80	70	58	71	76	72	31
73	85	74	69	75	100	76	49	77	96	78	64	79	76	80	25
81	80	82	69	83	105	84	55	85	114	86	80	87	100	88	38
89	96	90	75	91	105	92	49	93	91	94	58	95	65	96	18
97	65	98	58	99	91	100	49	101	105	102	75	103	96	104	38
105	100	106	80	107	114	108	55	109	105	110	69	111	80	112	25
113	76	114	64	115	96	116	49	117	100	118	69	119	85	120	31
121	76	122	58	123	80	124	36	125	65	126	40	127	43	128	10
129	45	130	43	131	70	132	40	133	88	134	65	135	85	136	36
137	97	138	80	139	115	140	58	141	112	142	76	143	90	144	31
145	97	146	85	147	128	148	69	149	140	150	100	151	124	152	49
153	121	154	96	155	133	156	64	157	117	158	76	159	85	160	25
161	88	162	80	163	124	164	69	165	144	166	105	167	133	168	55
169	140	170	114	171	160	172	80	173	149	174	100	175	115	176	38
177	112	178	96	179	142	180	75	181	149	182	105	183	128	184	49
185	117	186	91	187	124	188	58	189	103	190	65	191	70	192	18
193	70	194	65	195	103	196	58	197	124	198	91	199	117	200	49
201	128	202	105	203	149	204	75	205	142	206	96	207	112	208	38
209	115	210	100	211	149	212	80	213	160	214	114	215	140	216	55
217	133	218	105	219	144	220	69	221	124	222	80	223	88	224	25
225	85	226	76	227	117	228	64	229	133	230	96	231	121	232	49
233	124	234	100	235	140	236	69	237	128	238	85	239	97	240	31
241	90	242	76	243	112	244	58	245	115	246	80	247	97	248	36
249	85	250	65	251	88	252	40	253	70	254	43	255	45	256	10
257	46	258	45	259	73	260	43	261	93	262	70	263	91	264	40
265	105	266	88	267	125	268	65	269	123	270	85	271	100	272	36
273	109	274	97	275	144	276	80	277	158	278	115	279	141	280	58
281	139	282	112	283	153	284	76	285	136	286	90	287	100	288	31
289	105	290	97	291	148	292	85	293	172	294	128	295	160	296	69
297	169	298	140	299	193	300	100	301	181	302	124	303	141	304	49
305	139	306	121	307	176	308	96	309	185	310	133	311	160	312	64
313	148	314	117	315	157	316	76	317	132	318	85	319	91	320	25
321	93	322	88	323	137	324	80	325	165	326	124	327	157	328	69
329	172	330	144	331	200	332	105	333	192	334	133	335	153	336	55
337	158	338	140	339	204	340	114	341	219	342	160	343	193	344	80

Table B.8: *The Abelian complexity $\mathcal{AC}(n)$ of the 10-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
345	185	346	149	347	200	348	100	349	174	350	115	351	125	352	38
353	123	354	112	355	169	356	96	357	192	358	142	359	176	360	75
361	181	362	149	363	204	364	105	365	188	366	128	367	144	368	49
369	136	370	117	371	169	372	91	373	174	374	124	375	148	376	58
377	132	378	103	379	137	380	65	381	111	382	70	383	73	384	18
385	73	386	70	387	111	388	65	389	137	390	103	391	132	392	58
393	148	394	124	395	174	396	91	397	169	398	117	399	136	400	49
401	144	402	128	403	188	404	105	405	204	406	149	407	181	408	75
409	176	410	142	411	192	412	96	413	169	414	112	415	123	416	38
417	125	418	115	419	174	420	100	421	200	422	149	423	185	424	80
425	193	426	160	427	219	428	114	429	204	430	140	431	158	432	55
433	153	434	133	435	192	436	105	437	200	438	144	439	172	440	69
441	157	442	124	443	165	444	80	445	137	446	88	447	93	448	25
449	91	450	85	451	132	452	76	453	157	454	117	455	148	456	64
457	160	458	133	459	185	460	96	461	176	462	121	463	139	464	49
465	141	466	124	467	181	468	100	469	193	470	140	471	169	472	69
473	160	474	128	475	172	476	85	477	148	478	97	479	105	480	31
481	100	482	90	483	136	484	76	485	153	486	112	487	139	488	58
489	141	490	115	491	158	492	80	493	144	494	97	495	109	496	36
497	100	498	85	499	123	500	65	501	125	502	88	503	105	504	40
505	91	506	70	507	93	508	43	509	73	510	45	511	46	512	10
513	46	514	46	515	74	516	45	517	95	518	73	519	94	520	43
521	109	522	93	523	130	524	70	525	129	526	91	527	106	528	40
529	116	530	105	531	153	532	88	533	168	534	125	535	151	536	65
537	150	538	123	539	165	540	85	541	148	542	100	543	110	544	36
545	116	546	109	547	163	548	97	549	189	550	144	551	177	552	80
553	187	554	158	555	213	556	115	557	201	558	141	559	158	560	58
561	157	562	139	563	198	564	112	565	208	566	153	567	181	568	76
569	169	570	136	571	179	572	90	573	152	574	100	575	106	576	31
577	109	578	105	579	160	580	97	581	192	582	148	583	184	584	85
585	201	586	172	587	233	588	128	589	225	590	160	591	181	592	69
593	187	594	169	595	240	596	140	597	257	598	193	599	228	600	100
601	220	602	181	603	237	604	124	605	208	606	141	607	151	608	49
609	150	610	139	611	205	612	121	613	232	614	176	615	214	616	96
617	220	618	185	619	247	620	133	621	229	622	160	623	177	624	64
625	169	626	148	627	209	628	117	629	215	630	157	631	184	632	76
633	166	634	132	635	172	636	85	637	141	638	91	639	94	640	25
641	95	642	93	643	144	644	88	645	177	646	137	647	172	648	80
649	192	650	165	651	225	652	124	653	220	654	157	655	179	656	69
657	189	658	172	659	245	660	144	661	265	662	200	663	237	664	105
665	232	666	192	667	252	668	133	669	224	670	153	671	165	672	55
673	168	674	158	675	232	676	140	677	265	678	204	679	247	680	114
681	257	682	219	683	290	684	160	685	272	686	193	687	213	688	80

Table B.8: *The Abelian complexity $\mathcal{AC}(n)$ of the* 10*-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
689	208	690	185	691	259	692	149	693	269	694	200	695	233	696	100
697	215	698	174	699	225	700	115	701	189	702	125	703	130	704	38
705	129	706	123	707	186	708	112	709	220	710	169	711	209	712	96
713	225	714	192	715	259	716	142	717	248	718	176	719	198	720	75
721	201	722	181	723	256	724	149	725	272	726	204	727	240	728	105
729	229	730	188	731	245	732	128	733	213	734	144	735	153	736	49
737	148	738	136	739	200	740	117	741	224	742	169	743	205	744	91
745	208	746	174	747	232	748	124	749	213	750	148	751	163	752	58
753	152	754	132	755	186	756	103	757	189	758	137	759	160	760	65
761	141	762	111	763	144	764	70	765	115	766	73	767	74	768	18
769	74	770	73	771	115	772	70	773	144	774	111	775	141	776	65
777	160	778	137	779	189	780	103	781	186	782	132	783	152	784	58
785	163	786	148	787	213	788	124	789	232	790	174	791	208	792	91
793	205	794	169	795	224	796	117	797	200	798	136	799	148	800	49
801	153	802	144	803	213	804	128	805	245	806	188	807	229	808	105
809	240	810	204	811	272	812	149	813	256	814	181	815	201	816	75
817	198	818	176	819	248	820	142	821	259	822	192	823	225	824	96
825	209	826	169	827	220	828	112	829	186	830	123	831	129	832	38
833	130	834	125	835	189	836	115	837	225	838	174	839	215	840	100
841	233	842	200	843	269	844	149	845	259	846	185	847	208	848	80
849	213	850	193	851	272	852	160	853	290	854	219	855	257	856	114
857	247	858	204	859	265	860	140	861	232	862	158	863	168	864	55
865	165	866	153	867	224	868	133	869	252	870	192	871	232	872	105
873	237	874	200	875	265	876	144	877	245	878	172	879	189	880	69
881	179	882	157	883	220	884	124	885	225	886	165	887	192	888	80
889	172	890	137	891	177	892	88	893	144	894	93	895	95	896	25
897	94	898	91	899	141	900	85	901	172	902	132	903	166	904	76
905	184	906	157	907	215	908	117	909	209	910	148	911	169	912	64
913	177	914	160	915	229	916	133	917	247	918	185	919	220	920	96
921	214	922	176	923	232	924	121	925	205	926	139	927	150	928	49
929	151	930	141	931	208	932	124	933	237	934	181	935	220	936	100
937	228	938	193	939	257	940	140	941	240	942	169	943	187	944	69
945	181	946	160	947	225	948	128	949	233	950	172	951	201	952	85
953	184	954	148	955	192	956	97	957	160	958	105	959	109	960	31
961	106	962	100	963	152	964	90	965	179	966	136	967	169	968	76
969	181	970	153	971	208	972	112	973	198	974	139	975	157	976	58
977	158	978	141	979	201	980	115	981	213	982	158	983	187	984	80
985	177	986	144	987	189	988	97	989	163	990	109	991	116	992	36
993	110	994	100	995	148	996	85	997	165	998	123	999	150	1000	65

Table B.8: *The Abelian complexity $\mathcal{AC}(n)$ of the 10-bonacci word for $n \in \{1, \ldots, 1000\}$.*

B.9 11-bonacci word

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
1	11	2	11	3	20	4	11	5	28	6	20	7	28	8	11
9	35	10	28	11	43	12	20	13	43	14	28	15	35	16	11
17	41	18	35	19	56	20	28	21	63	22	43	23	56	24	20
25	56	26	43	27	63	28	28	29	56	30	35	31	41	32	11
33	46	34	41	35	67	36	35	37	80	38	56	39	74	40	28
41	80	42	63	43	93	44	43	45	87	46	56	47	67	48	20
49	67	50	56	51	87	52	43	53	93	54	63	55	80	56	28
57	74	58	56	59	80	60	35	61	67	62	41	63	46	64	11
65	50	66	46	67	76	68	41	69	94	70	67	71	89	72	35
73	100	74	80	75	118	76	56	77	113	78	74	79	89	80	28
81	94	82	80	83	124	84	63	85	135	86	93	87	118	88	43
89	113	90	87	91	124	92	56	93	107	94	67	95	76	96	20
97	76	98	67	99	107	100	56	101	124	102	87	103	113	104	43
105	118	106	93	107	135	108	63	109	124	110	80	111	94	112	28
113	89	114	74	115	113	116	56	117	118	118	80	119	100	120	35
121	89	122	67	123	94	124	41	125	76	126	46	127	50	128	11
129	53	130	50	131	83	132	46	133	105	134	76	135	101	136	41
137	116	138	94	139	138	140	67	141	134	142	89	143	107	144	35
145	116	146	100	147	154	148	80	149	169	150	118	151	149	152	56
153	145	154	113	155	160	156	74	157	140	158	89	159	101	160	28
161	105	162	94	163	149	164	80	165	174	166	124	167	160	168	63
169	169	170	135	171	194	172	93	173	180	174	118	175	138	176	43
177	134	178	113	179	171	180	87	181	180	182	124	183	154	184	56
185	140	186	107	187	149	188	67	189	123	190	76	191	83	192	20
193	83	194	76	195	123	196	67	197	149	198	107	199	140	200	56
201	154	202	124	203	180	204	87	205	171	206	113	207	134	208	43
209	138	210	118	211	180	212	93	213	194	214	135	215	169	216	63
217	160	218	124	219	174	220	80	221	149	222	94	223	105	224	28
225	101	226	89	227	140	228	74	229	160	230	113	231	145	232	56
233	149	234	118	235	169	236	80	237	154	238	100	239	116	240	35
241	107	242	89	243	134	244	67	245	138	246	94	247	116	248	41
249	101	250	76	251	105	252	46	253	83	254	50	255	53	256	11
257	55	258	53	259	88	260	50	261	113	262	83	263	110	264	46
265	128	266	105	267	153	268	76	269	150	270	101	271	121	272	41
273	133	274	116	275	177	276	94	277	195	278	138	279	173	280	67
281	170	282	134	283	188	284	89	285	166	286	107	287	121	288	35
289	128	290	116	291	182	292	100	293	213	294	154	295	197	296	80
297	209	298	169	299	240	300	118	301	224	302	149	303	173	304	56
305	170	306	145	307	217	308	113	309	229	310	160	311	197	312	74
313	181	314	140	315	193	316	89	317	161	318	101	319	110	320	28
321	113	322	105	323	168	324	94	325	204	326	149	327	193	328	80
329	213	330	174	331	249	332	124	333	238	334	160	335	188	336	63
337	195	338	169	339	254	340	135	341	274	342	194	343	240	344	93

Table B.9: *The Abelian complexity $\mathcal{AC}(n)$ of the 11-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
345	229	346	180	347	249	348	118	349	215	350	138	351	153	352	43
353	150	354	134	355	208	356	113	357	238	358	171	359	217	360	87
361	224	362	180	363	254	364	124	365	233	366	154	367	177	368	56
369	166	370	140	371	208	372	107	373	215	374	149	375	182	376	67
377	161	378	123	379	168	380	76	381	135	382	83	383	88	384	20
385	88	386	83	387	135	388	76	389	168	390	123	391	161	392	67
393	182	394	149	395	215	396	107	397	208	398	140	399	166	400	56
401	177	402	154	403	233	404	124	405	254	406	180	407	224	408	87
409	217	410	171	411	238	412	113	413	208	414	134	415	150	416	43
417	153	418	138	419	215	420	118	421	249	422	180	423	229	424	93
425	240	426	194	427	274	428	135	429	254	430	169	431	195	432	63
433	188	434	160	435	238	436	124	437	249	438	174	439	213	440	80
441	193	442	149	443	204	444	94	445	168	446	105	447	113	448	28
449	110	450	101	451	161	452	89	453	193	454	140	455	181	456	74
457	197	458	160	459	229	460	113	461	217	462	145	463	170	464	56
465	173	466	149	467	224	468	118	469	240	470	169	471	209	472	80
473	197	474	154	475	213	476	100	477	182	478	116	479	128	480	35
481	121	482	107	483	166	484	89	485	188	486	134	487	170	488	67
489	173	490	138	491	195	492	94	493	177	494	116	495	133	496	41
497	121	498	101	499	150	500	76	501	153	502	105	503	128	504	46
505	110	506	83	507	113	508	50	509	88	510	53	511	55	512	11
513	56	514	55	515	91	516	53	517	118	518	88	519	116	520	50
521	136	522	113	523	163	524	83	525	161	526	110	527	131	528	46
529	145	530	128	531	193	532	105	533	213	534	153	535	190	536	76
537	188	538	150	539	208	540	101	541	185	542	121	543	136	544	41
545	145	546	133	547	206	548	116	549	241	550	177	551	224	552	94
553	238	554	195	555	273	556	138	557	256	558	173	559	199	560	67
561	197	562	170	563	251	564	134	565	265	566	188	567	229	568	89
569	212	570	166	571	226	572	107	573	190	574	121	575	131	576	35
577	136	578	128	579	202	580	116	581	245	582	182	583	233	584	100
585	257	586	213	587	300	588	154	589	288	590	197	591	229	592	80
593	238	594	209	595	309	596	169	597	333	598	240	599	293	600	118
601	281	602	224	603	305	604	149	605	265	606	173	607	190	608	56
609	188	610	170	611	260	612	145	613	297	614	217	615	272	616	113
617	281	618	229	619	318	620	160	621	293	622	197	623	224	624	74
625	212	626	181	627	265	628	140	629	274	630	193	631	233	632	89
633	208	634	161	635	217	636	101	637	176	638	110	639	116	640	28
641	118	642	113	643	181	644	105	645	225	646	168	647	217	648	94
649	245	650	204	651	289	652	149	653	281	654	193	655	226	656	80
657	241	658	213	659	316	660	174	661	344	662	249	663	305	664	124
665	297	666	238	667	325	668	160	669	286	670	188	671	208	672	63
673	213	674	195	675	298	676	169	677	344	678	254	679	318	680	135
681	333	682	274	683	379	684	194	685	353	686	240	687	273	688	93

***Table B.9:** The Abelian complexity $\mathcal{AC}(n)$ of the 11-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
689	265	690	229	691	334	692	180	693	349	694	249	695	300	696	118
697	274	698	215	699	289	700	138	701	240	702	153	703	163	704	43
705	161	706	150	707	235	708	134	709	281	710	208	711	265	712	113
713	288	714	238	715	334	716	171	717	318	718	217	719	251	720	87
721	256	722	224	723	330	724	180	725	353	726	254	727	309	728	124
729	293	730	233	731	316	732	154	733	272	734	177	735	193	736	56
737	185	738	166	739	253	740	140	741	286	742	208	743	260	744	107
745	265	746	215	747	298	748	149	749	272	750	182	751	206	752	67
753	190	754	161	755	235	756	123	757	240	758	168	759	202	760	76
761	176	762	135	763	181	764	83	765	143	766	88	767	91	768	20
769	91	770	88	771	143	772	83	773	181	774	135	775	176	776	76
777	202	778	168	779	240	780	123	781	235	782	161	783	190	784	67
785	206	786	182	787	272	788	149	789	298	790	215	791	265	792	107
793	260	794	208	795	286	796	140	797	253	798	166	799	185	800	56
801	193	802	177	803	272	804	154	805	316	806	233	807	293	808	124
809	309	810	254	811	353	812	180	813	330	814	224	815	256	816	87
817	251	818	217	819	318	820	171	821	334	822	238	823	288	824	113
825	265	826	208	827	281	828	134	829	235	830	150	831	161	832	43
833	163	834	153	835	240	836	138	837	289	838	215	839	274	840	118
841	300	842	249	843	349	844	180	845	334	846	229	847	265	848	93
849	273	850	240	851	353	852	194	853	379	854	274	855	333	856	135
857	318	858	254	859	344	860	169	861	298	862	195	863	213	864	63
865	208	866	188	867	286	868	160	869	325	870	238	871	297	872	124
873	305	874	249	875	344	876	174	877	316	878	213	879	241	880	80
881	226	882	193	883	281	884	149	885	289	886	204	887	245	888	94
889	217	890	168	891	225	892	105	893	181	894	113	895	118	896	28
897	116	898	110	899	176	900	101	901	217	902	161	903	208	904	89
905	233	906	193	907	274	908	140	909	265	910	181	911	212	912	74
913	224	914	197	915	293	916	160	917	318	918	229	919	281	920	113
921	272	922	217	923	297	924	145	925	260	926	170	927	188	928	56
929	190	930	173	931	265	932	149	933	305	934	224	935	281	936	118
937	293	938	240	939	333	940	169	941	309	942	209	943	238	944	80
945	229	946	197	947	288	948	154	949	300	950	213	951	257	952	100
953	233	954	182	955	245	956	116	957	202	958	128	959	136	960	35
961	131	962	121	963	190	964	107	965	226	966	166	967	212	968	89
969	229	970	188	971	265	972	134	973	251	974	170	975	197	976	67
977	199	978	173	979	256	980	138	981	273	982	195	983	238	984	94
985	224	986	177	987	241	988	116	989	206	990	133	991	145	992	41
993	136	994	121	995	185	996	101	997	208	998	150	999	188	1000	76

Table B.9: *The Abelian complexity $\mathcal{AC}(n)$ of the* 11*-bonacci word for* $n \in \{1, \ldots, 1000\}$.

B.10 12-bonacci word

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
1	12	2	12	3	22	4	12	5	31	6	22	7	31	8	12
9	39	10	31	11	48	12	22	13	48	14	31	15	39	16	12
17	46	18	39	19	63	20	31	21	71	22	48	23	63	24	22
25	63	26	48	27	71	28	31	29	63	30	39	31	46	32	12
33	52	34	46	35	76	36	39	37	91	38	63	39	84	40	31
41	91	42	71	43	106	44	48	45	99	46	63	47	76	48	22
49	76	50	63	51	99	52	48	53	106	54	71	55	91	56	31
57	84	58	63	59	91	60	39	61	76	62	46	63	52	64	12
65	57	66	52	67	87	68	46	69	108	70	76	71	102	72	39
73	115	74	91	75	136	76	63	77	130	78	84	79	102	80	31
81	108	82	91	83	143	84	71	85	156	86	106	87	136	88	48
89	130	90	99	91	143	92	63	93	123	94	76	95	87	96	22
97	87	98	76	99	123	100	63	101	143	102	99	103	130	104	48
105	136	106	106	107	156	108	71	109	143	110	91	111	108	112	31
113	102	114	84	115	130	116	63	117	136	118	91	119	115	120	39
121	102	122	76	123	108	124	46	125	87	126	52	127	57	128	12
129	61	130	57	131	96	132	52	133	122	134	87	135	117	136	46
137	135	138	108	139	161	140	76	141	156	142	102	143	124	144	39
145	135	146	115	147	180	148	91	149	198	150	136	151	174	152	63
153	169	154	130	155	187	156	84	157	163	158	102	159	117	160	31
161	122	162	108	163	174	164	91	165	204	166	143	167	187	168	71
169	198	170	156	171	228	172	106	173	211	174	136	175	161	176	48
177	156	178	130	179	200	180	99	181	211	182	143	183	180	184	63
185	163	186	123	187	174	188	76	189	143	190	87	191	96	192	22
193	96	194	87	195	143	196	76	197	174	198	123	199	163	200	63
201	180	202	143	203	211	204	99	205	200	206	130	207	156	208	48
209	161	210	136	211	211	212	106	213	228	214	156	215	198	216	71
217	187	218	143	219	204	220	91	221	174	222	108	223	122	224	31
225	117	226	102	227	163	228	84	229	187	230	130	231	169	232	63
233	174	234	136	235	198	236	91	237	180	238	115	239	135	240	39
241	124	242	102	243	156	244	76	245	161	246	108	247	135	248	46
249	117	250	87	251	122	252	52	253	96	254	57	255	61	256	12
257	64	258	61	259	103	260	57	261	133	262	96	263	129	264	52
265	151	266	122	267	181	268	87	269	177	270	117	271	142	272	46
273	157	274	135	275	210	276	108	277	232	278	161	279	205	280	76
281	201	282	156	283	223	284	102	285	196	286	124	287	142	288	39
289	151	290	135	291	216	292	115	293	254	294	180	295	234	296	91
297	249	298	198	299	287	300	136	301	267	302	174	303	205	304	63
305	201	306	169	307	258	308	130	309	273	310	187	311	234	312	84
313	214	314	163	315	229	316	102	317	190	318	117	319	129	320	31
321	133	322	122	323	199	324	108	325	243	326	174	327	229	328	91
329	254	330	204	331	298	332	143	333	284	334	187	335	223	336	71
337	232	338	198	339	304	340	156	341	329	342	228	343	287	344	106

Table B.10: *The Abelian complexity $\mathcal{AC}(n)$ of the 12-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
345	273	346	211	347	298	348	136	349	256	350	161	351	181	352	48
353	177	354	156	355	247	356	130	357	284	358	200	359	258	360	99
361	267	362	211	363	304	364	143	365	278	366	180	367	210	368	63
369	196	370	163	371	247	372	123	373	256	374	174	375	216	376	76
377	190	378	143	379	199	380	87	381	159	382	96	383	103	384	22
385	103	386	96	387	159	388	87	389	199	390	143	391	190	392	76
393	216	394	174	395	256	396	123	397	247	398	163	399	196	400	63
401	210	402	180	403	278	404	143	405	304	406	211	407	267	408	99
409	258	410	200	411	284	412	130	413	247	414	156	415	177	416	48
417	181	418	161	419	256	420	136	421	298	422	211	423	273	424	106
425	287	426	228	427	329	428	156	429	304	430	198	431	232	432	71
433	223	434	187	435	284	436	143	437	298	438	204	439	254	440	91
441	229	442	174	443	243	444	108	445	199	446	122	447	133	448	31
449	129	450	117	451	190	452	102	453	229	454	163	455	214	456	84
457	234	458	187	459	273	460	130	461	258	462	169	463	201	464	63
465	205	466	174	467	267	468	136	469	287	470	198	471	249	472	91
473	234	474	180	475	254	476	115	477	216	478	135	479	151	480	39
481	142	482	124	483	196	484	102	485	223	486	156	487	201	488	76
489	205	490	161	491	232	492	108	493	210	494	135	495	157	496	46
497	142	498	117	499	177	500	87	501	181	502	122	503	151	504	52
505	129	506	96	507	133	508	57	509	103	510	61	511	64	512	12
513	66	514	64	515	108	516	61	517	141	518	103	519	138	520	57
521	163	522	133	523	196	524	96	525	193	526	129	527	156	528	52
529	174	530	151	531	233	532	122	533	258	534	181	535	229	536	87
537	226	538	177	539	251	540	117	541	222	542	142	543	162	544	46
545	174	546	157	547	249	548	135	549	293	550	210	551	271	552	108
553	289	554	232	555	333	556	161	557	311	558	205	559	240	560	76
561	237	562	201	563	304	564	156	565	322	566	223	567	277	568	102
569	255	570	196	571	273	572	124	573	228	574	142	575	156	576	39
577	163	578	151	579	244	580	135	581	298	582	216	583	282	584	115
585	313	586	254	587	367	588	180	589	351	590	234	591	277	592	91
593	289	594	249	595	378	596	198	597	409	598	287	599	358	600	136
601	342	602	267	603	373	604	174	605	322	606	205	607	229	608	63
609	226	610	201	611	315	612	169	613	362	614	258	615	330	616	130
617	342	618	273	619	389	620	187	621	357	622	234	623	271	624	84
625	255	626	214	627	321	628	163	629	333	630	229	631	282	632	102
633	250	634	190	635	262	636	117	637	211	638	129	639	138	640	31
641	141	642	133	643	218	644	122	645	273	646	199	647	262	648	108
649	298	650	243	651	353	652	174	653	342	654	229	655	273	656	91
657	293	658	254	659	387	660	204	661	423	662	298	663	373	664	143
665	362	666	284	667	398	668	187	669	348	670	223	671	251	672	71
673	258	674	232	675	364	676	198	677	423	678	304	679	389	680	156
681	409	682	329	683	468	684	228	685	434	686	287	687	333	688	106

Table B.10: *The Abelian complexity $\mathcal{AC}(n)$ of the 12-bonacci word for $n \in \{1, \ldots, 1000\}$.*

n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$	n	$\mathcal{AC}(n)$
689	322	690	273	691	409	692	211	693	429	694	298	695	367	696	136
697	333	698	256	699	353	700	161	701	291	702	181	703	196	704	48
705	193	706	177	707	284	708	156	709	342	710	247	711	321	712	130
713	351	714	284	715	409	716	200	717	388	718	258	719	304	720	99
721	311	722	267	723	404	724	211	725	434	726	304	727	378	728	143
729	357	730	278	731	387	732	180	733	331	734	210	735	233	736	63
737	222	738	196	739	306	740	163	741	348	742	247	743	315	744	123
745	322	746	256	747	364	748	174	749	331	750	216	751	249	752	76
753	228	754	190	755	284	756	143	757	291	758	199	759	244	760	87
761	211	762	159	763	218	764	96	765	171	766	103	767	108	768	22
769	108	770	103	771	171	772	96	773	218	774	159	775	211	776	87
777	244	778	199	779	291	780	143	781	284	782	190	783	228	784	76
785	249	786	216	787	331	788	174	789	364	790	256	791	322	792	123
793	315	794	247	795	348	796	163	797	306	798	196	799	222	800	63
801	233	802	210	803	331	804	180	805	387	806	278	807	357	808	143
809	378	810	304	811	434	812	211	813	404	814	267	815	311	816	99
817	304	818	258	819	388	820	200	821	409	822	284	823	351	824	130
825	321	826	247	827	342	828	156	829	284	830	177	831	193	832	48
833	196	834	181	835	291	836	161	837	353	838	256	839	333	840	136
841	367	842	298	843	429	844	211	845	409	846	273	847	322	848	106
849	333	850	287	851	434	852	228	853	468	854	329	855	409	856	156
857	389	858	304	859	423	860	198	861	364	862	232	863	258	864	71
865	251	866	223	867	348	868	187	869	398	870	284	871	362	872	143
873	373	874	298	875	423	876	204	877	387	878	254	879	293	880	91
881	273	882	229	883	342	884	174	885	353	886	243	887	298	888	108
889	262	890	199	891	273	892	122	893	218	894	133	895	141	896	31
897	138	898	129	899	211	900	117	901	262	902	190	903	250	904	102
905	282	906	229	907	333	908	163	909	321	910	214	911	255	912	84
913	271	914	234	915	357	916	187	917	389	918	273	919	342	920	130
921	330	922	258	923	362	924	169	925	315	926	201	927	226	928	63
929	229	930	205	931	322	932	174	933	373	934	267	935	342	936	136
937	358	938	287	939	409	940	198	941	378	942	249	943	289	944	91
945	277	946	234	947	351	948	180	949	367	950	254	951	313	952	115
953	282	954	216	955	298	956	135	957	244	958	151	959	163	960	39
961	156	962	142	963	228	964	124	965	273	966	196	967	255	968	102
969	277	970	223	971	322	972	156	973	304	974	201	975	237	976	76
977	240	978	205	979	311	980	161	981	333	982	232	983	289	984	108
985	271	986	210	987	293	988	135	989	249	990	157	991	174	992	46
993	162	994	142	995	222	996	117	997	251	998	177	999	226	1000	87

Table B.10: *The Abelian complexity $\mathcal{AC}(n)$ of the 12-bonacci word for $n \in \{1, \ldots, 1000\}$.*

APPENDIX C

Semi-Abelian returns in d-bonacci word

C.1 Fibonacci word

n	**Ab. class**	**#sAr**	**semi-Abelian returns (sAr)**
1	$(1, 0)$	2	0, 01
1	$(0, 1)$	2	10, 100
2	$(1, 1)$	3	1, 10, 0
2	$(2, 0)$	2	00101, 001
3	$(2, 1)$	3	1, 0, 01
3	$(1, 2)$	2	10100, 10100100
4	$(3, 1)$	3	0, 001, 01
4	$(2, 2)$	3	10, 0, 100
5	$(3, 2)$	3	10, 1, 0
5	$(4, 1)$	2	00100101, 0010010100101
6	$(4, 2)$	3	1, 0, 01
6	$(3, 3)$	3	10, 10100, 100
7	$(4, 3)$	3	1, 10, 0
7	$(5, 2)$	3	00101, 01, 001
8	$(5, 3)$	3	1, 0, 01
8	$(4, 4)$	2	101001010010010100100, 1010010100100
9	$(6, 3)$	3	0, 001, 01
9	$(5, 4)$	3	10, 0, 100
10	$(6, 4)$	3	1, 10, 0
10	$(7, 3)$	3	00101, 00100101, 001
11	$(7, 4)$	3	1, 0, 01
11	$(6, 5)$	3	10100, 10100100, 100
12	$(8, 4)$	3	0, 001, 01
12	$(7, 5)$	3	10, 0, 100
13	$(8, 5)$	3	10, 1, 0
13	$(9, 4)$	2	0010010100100101001010010010100101, 001001010010010100101
14	$(9, 5)$	3	1, 0, 01
14	$(8, 6)$	3	10, 10100, 100

Table C.1: *Semi-Abelian returns in the Fibonacci word*

n	Ab. class	#sAr	semi-Abelian returns (sAr)
15	$(9, 6)$	3	1, 10, 0
15	$(10, 5)$	3	00101, 01, 001
16	$(10, 6)$	3	1, 0, 01
16	$(9, 7)$	3	10100, 1010010100100, 10100100
17	$(11, 6)$	3	0, 001, 01
17	$(10, 7)$	3	10, 0, 100
18	$(11, 7)$	3	10, 1, 0
18	$(12, 6)$	3	00101, 0010010100101, 00100101
19	$(12, 7)$	3	1, 0, 01
19	$(11, 8)$	3	10, 10100, 100
20	$(12, 8)$	3	1, 10, 0
20	$(13, 7)$	3	00101, 01, 001
21	$(13, 8)$	3	1, 0, 01
21	$(12, 9)$	2	1010010100100101001010010010100100101001010010010100100, 1010 010100100101001010010010100100
22	$(14, 8)$	3	0, 001, 01
22	$(13, 9)$	3	10, 0, 100
23	$(14, 9)$	3	1, 10, 0
23	$(15, 8)$	3	00101, 00100101, 001
24	$(15, 9)$	3	1, 0, 01
24	$(14, 10)$	3	10100, 10100100, 100
25	$(16, 9)$	3	0, 001, 01
25	$(15, 10)$	3	10, 0, 100
26	$(16, 10)$	3	10, 1, 0
26	$(17, 9)$	3	00100101, 001001010010010100101, 0010010100101
27	$(17, 10)$	3	1, 0, 01
27	$(16, 11)$	3	10, 10100, 100
28	$(17, 11)$	3	1, 10, 0
28	$(18, 10)$	3	00101, 01, 001
29	$(18, 11)$	3	1, 0, 01
29	$(17, 12)$	3	101001010010010100100, 1010010100100, 10100100
30	$(19, 11)$	3	0, 001, 01
30	$(18, 12)$	3	10, 0, 100
31	$(19, 12)$	3	1, 10, 0
31	$(20, 11)$	3	00101, 00100101, 001
32	$(20, 12)$	3	1, 0, 01
32	$(19, 13)$	3	10100, 10100100, 100
33	$(21, 12)$	3	0, 001, 01
33	$(20, 13)$	3	10, 0, 100
34	$(21, 13)$	3	10, 1, 0
34	$(22, 12)$	2	001001010010010100101001001010010010100101001001010010100100 10100100101001010010010100101, 001001010010010100101001001010 0100101001010010010100101
35	$(22, 13)$	3	1, 0, 01
35	$(21, 14)$	3	10, 10100, 100

Table C.1: *Semi-Abelian returns in the Fibonacci word*

n	**Ab. class**	**#sAr**	**semi-Abelian returns (sAr)**
36	$(22, 14)$	3	1, 10, 0
36	$(23, 13)$	3	00101, 01, 001
37	$(23, 14)$	3	1, 0, 01
37	$(22, 15)$	3	10100, 1010010100100, 10100100
38	$(24, 14)$	3	0, 001, 01
38	$(23, 15)$	3	10, 0, 100
39	$(24, 15)$	3	10, 1, 0
39	$(25, 14)$	3	00101, 00100101, 0010010100101
40	$(25, 15)$	3	1, 0, 01
40	$(24, 16)$	3	10, 10100, 100
41	$(25, 16)$	3	1, 10, 0
41	$(26, 15)$	3	00101, 01, 001
42	$(26, 16)$	3	1, 0, 01
42	$(25, 17)$	3	101001010010010100100, 1010010100100, 101001010010010100101001 0010100100
43	$(27, 16)$	3	0, 001, 01
43	$(26, 17)$	3	10, 0, 100
44	$(27, 17)$	3	1, 10, 0
44	$(28, 16)$	3	00101, 00100101, 001
45	$(28, 17)$	3	1, 0, 01
45	$(27, 18)$	3	10100, 10100100, 100
46	$(29, 17)$	3	0, 001, 01
46	$(28, 18)$	3	10, 0, 100
47	$(29, 18)$	3	10, 1, 0
47	$(30, 17)$	3	0010010100100101001010010010100101, 0010010100101, 00100101001 0010100101
48	$(30, 18)$	3	1, 0, 01
48	$(29, 19)$	3	10, 10100, 100
49	$(30, 19)$	3	1, 10, 0
49	$(31, 18)$	3	00101, 01, 001
50	$(31, 19)$	3	1, 0, 01
50	$(30, 20)$	3	10100, 1010010100100, 10100100

Table C.1: *Semi-Abelian returns in the Fibonacci word*

C.2 Tribonacci word

n	Ab. class	#sAr	semi-Abelian returns (sAr)
1	$(1, 0, 0)$	3	0, 01, 02
1	$(0, 1, 0)$	3	1020, 10, 100
1	$(0, 0, 1)$	3	2010010, 2010, 201010
2	$(1, 1, 0)$	4	1, 10, 0, 102
2	$(1, 0, 1)$	4	0, 20101, 201, 201001
2	$(2, 0, 0)$	3	0010201, 0010201010201, 00102010201
3	$(2, 1, 0)$	4	1, 0, 01, 0102
3	$(1, 1, 1)$	4	20, 20100, 10, 2010
3	$(2, 0, 1)$	3	0201, 020101, 0201001
3	$(1, 2, 0)$	3	10102010010201001020, 1010201001020102010001020, 1010201001020
4	$(2, 1, 1)$	5	2, 1, 0, 201, 2010
4	$(3, 1, 0)$	4	00102010102, 00102, 001020102, 01
4	$(2, 2, 0)$	5	10010201020, 0, 100102, 1001020, 101020
5	$(3, 1, 1)$	5	0201, 0, 201, 01, 02
5	$(2, 2, 1)$	4	1020, 20, 10, 1020100
5	$(3, 2, 0)$	4	010102, 0, 100102, 1001020102
5	$(2, 1, 2)$	3	201020100102010102010010201001020101020100100, 201020100102010 102010010201010201001 0, 20102010010201010201001 0
6	$(3, 2, 1)$	6	2, 20, 1, 10, 0, 102
6	$(4, 1, 1)$	4	0201, 0010201, 001020101, 001
6	$(4, 2, 0)$	3	0100102, 01001020102, 0100102010102
6	$(3, 1, 2)$	4	201020100102010102010010201010201001, 0, 201020100102010102010 010201001020101020100 1, 201020100102010102010 01
7	$(4, 2, 1)$	6	2, 1, 0, 0102, 01, 02
7	$(3, 3, 1)$	4	1020, 10102010010201020100, 1010201001020100, 101020100
7	$(3, 2, 2)$	5	10, 2010102010010201001 0, 20101020100, 201010201001 0, 2010201001 0
7	$(4, 1, 2)$	3	02010201001020101020100102010010201010201001, 020102010010201 010201001, 0201020100102010102010010201010201001
8	$(5, 2, 1)$	5	0, 0010201, 00102, 01, 02
8	$(4, 3, 1)$	6	1020, 10, 0, 1001020, 102, 100
8	$(4, 2, 2)$	6	1, 0, 2010010, 2010, 201010, 201001
9	$(5, 3, 1)$	6	10, 1, 0, 100102, 102, 0102
9	$(5, 2, 2)$	5	020101, 0, 201, 201001, 01
9	$(4, 3, 2)$	4	10, 20101020100, 10201020100, 201010201001020100
9	$(6, 2, 1)$	3	0010201001020101020100102010201001020101020100102010102 01001 020102010010201010201, 0010201001020101020100102010201001 0201 010201, 00102010010201010201001020102010010201010201001020102 010010201010201
10	$(6, 3, 1)$	6	1, 0, 0100102, 00102, 01, 0102
10	$(5, 3, 2)$	6	20100, 20, 1, 10, 0, 2010
10	$(6, 2, 2)$	4	0201, 0201001020101, 001, 0201001

Table C.2: *Semi-Abelian returns in the Tribonacci word*

n	Ab. class	#sAr	semi-Abelian returns (sAr)
10	$(5,4,1)$	4	10102010010201020, 1001020, 101020, 1010201001020
11	$(6,3,2)$	6	2, 1, 0, 201, 2010, 01
11	$(6,4,1)$	5	10102, 0, 1010201001020102, 100102, 101020
11	$(5,4,2)$	3	102010102010010201020100, 10201010201001020100, 1020101020100
11	$(7,3,1)$	4	001020100102010102010010201020100102010102, 00102010010201010201001020102010010201020100102010102010010201020100102010102, 0010201001020101020100102010201001020101020100102010102010010201020100102010102, 01
12	$(7,3,2)$	6	0201, 0, 201, 001, 01, 02
12	$(6,4,2)$	6	1020, 20, 1020100, 10, 0, 100
12	$(7,4,1)$	5	010102, 0, 100102, 0100102, 01010201001020102
12	$(6,3,3)$	4	20102010010201010, 2010201001020101020100102010010201010, 201020100102010102010010201010, 2010010
13	$(7,4,2)$	6	20, 2, 10, 1, 0, 102
13	$(7,3,3)$	5	201020100102010102010010201001020101, 2010201001020101, 0, 20102010010201010201001020101, 201001
13	$(8,3,2)$	5	0201, 00102010010201010201, 00102010201001020101, 00102010201001020101020100102010201, 001020102010010201010201
13	$(8,4,1)$	3	01001020100102010102010010201020100102010102, 010010201001020101020100102010201001020101020100102010102010010201020100102010102, 01001020100102010102010010201020100102010102010010201020100102010102
14	$(8,4,2)$	6	2, 1, 0, 0102, 01, 02
14	$(7,5,2)$	5	1020, 10, 101020100, 1010201001020, 100
14	$(7,4,3)$	6	20100, 10, 2010010, 2010102010010, 2010, 201010
14	$(8,3,3)$	4	0201020100102010102010010201001020101, 020102010010201010201001020101, 02010201001020101, 0201001
15	$(8,5,2)$	7	1020, 1, 10, 0, 100102, 101020, 102
15	$(8,4,3)$	7	1, 0, 20101, 201, 2010, 201010, 201001
15	$(9,4,2)$	6	00102010102, 0010201, 0010201010201, 01, 02, 00102010201
16	$(9,5,2)$	7	010102, 1, 0, 100102, 102, 01, 0102
16	$(8,5,3)$	6	20, 20100, 1020100, 10, 20101020100, 2010
16	$(9,4,3)$	7	020101, 0201, 0, 201, 201001, 01, 0201001
16	$(8,6,2)$	3	1010201001020101020100102010201001020101020100102010010201010201001020102010010201010201001020102010010201010201001020101020100102010201001020, 10102010010201010201001020102010010201010201001020100102010102010010201020100102010102010010201001020101020100102010201001020101020100102010201001020101020100102010201001020, 101020100102010102010010201020100102010102010010201001020101020100102010201001020
17	$(9,5,3)$	7	20, 2, 10, 1, 0, 201, 2010
17	$(10,5,2)$	5	00102010102, 01001020102, 00102, 0102, 01
17	$(9,6,2)$	5	0, 100102, 100102010102010010201020, 1001020, 101020
17	$(10,4,3)$	4	02010010201001020101, 0201, 00102010201001020101, 001020102010010201010201001020101
18	$(10,5,3)$	7	0201, 2, 1, 0, 201, 01, 02

Table C.2: *Semi-Abelian returns in the Tribonacci word*

n	Ab. class	#sAr	semi-Abelian returns (sAr)
18	$(9,6,3)$	6	20, 1020, 10, 1020100, 1020101020100, 101020100
18	$(10,6,2)$	5	010102, 10102, 0, 100102, 10010201010201001020102
18	$(9,5,4)$	4	2010201001020101020100102010010, 2010102010010, 20102010010, 201020100102010102010010
19	$(10,6,3)$	7	2, 1020, 20, 1, 10, 0, 102
19	$(11,5,3)$	6	0201, 0010201, 001020101, 001, 01, 02
19	$(11,6,2)$	4	010102, 0100102, 010010201010201001020102, 0100102010102
19	$(10,5,4)$	6	201020100102010102010010201001, 0, 20102010010201010201001, 2010201001, 201001, 201010
20	$(11,6,3)$	7	2, 1, 0, 102, 01, 0102, 02
20	$(10,6,4)$	5	10, 20101020100, 20102010010, 201020100, 20102010010201010201001020100
20	$(11,5,4)$	6	020101, 0, 020102010010201010201001, 201001, 0201020100102010102010010201001, 02010201001
20	$(10,7,3)$	4	1020, 10102010010201010201001020102010010201010201001020100102010102010010201020100, 10102010010201010201001020102010010201010201001020100102010102010010201020100102010102010010201020100, 10102010010201010201001020102010010201010201001020102010010201010201001020102010010201010201001020102010010201010201001020102010010201010201001020102010010201020100
21	$(12,6,3)$	6	0010201, 0, 00102, 0102, 01, 02
21	$(11,7,3)$	7	1020, 10, 0, 1001020, 101020, 102, 100
21	$(11,6,4)$	7	10, 1, 0, 2010010, 2010, 201001, 201010
21	$(12,5,4)$	3	0201001020102010010201010201001020101, 02010010201020100102010102010010201001020101, 020100102010201001020101
22	$(12,7,3)$	7	1, 10, 10102, 0, 100102, 102, 0102
22	$(12,6,4)$	7	020101, 1, 0, 20101, 201, 201001, 01
22	$(11,7,4)$	5	10, 20101020100, 10201020100, 1020101020100, 1020102010010201010201001020100
22	$(13,6,3)$	4	0010201001020101020100102010201001020101020100102010201, 0010201010201, 00102010010201010201001020102010010201010201001020101020100102010201, 0010201001020101020100102010201
23	$(13,7,3)$	7	1, 010102, 0, 0100102, 00102, 01, 0102
23	$(12,7,4)$	6	20100, 20, 1, 10, 0, 2010
23	$(13,6,4)$	6	020101, 0201, 0201001020101, 01, 001, 0201001
23	$(12,8,3)$	5	10102010010201001020101020100102010201001020101020100102010201001020, 1001020, 10102010010201001020101020100102010201001020, 1010201001020100102010102010010201020, 1010201001020101020100102010201001020
24	$(13,7,4)$	6	2, 1, 0, 201, 2010, 01
24	$(14,7,3)$	5	00102010102, 00102010010201010201001020102010010201010201001020102, 001020100102010102010010201020100102010102010010201010201001020102, 01, 00102010010201010201001020102

Table C.2: Semi-Abelian returns in the Tribonacci word

n	Ab. class	#sAr	semi-Abelian returns (sAr)
24	$(13, 8, 3)$	7	100102010010201010201001020102, 10102010010201010201001020102 0, 0, 100102, 101020, 100102010010201010201001020102010010201010 2010010201020, 1001020100102010102010010201020
24	$(12, 8, 4)$	3	102010102010010201010201001020102010010201010201001020100102 010102010010201020100102010102010010201020100102010102010010 20100102010102010010201020100, 102010102010010201010201001020 102010010201010201001020100102010102010010201020100, 10201010 201001020101020100102010201001020101020100102010010201010201 001020102010010201010201001020100102010102010010201020100
25	$(14, 7, 4)$	6	0201, 0, 201, 001, 01, 02
25	$(13, 8, 4)$	6	20, 1020, 1020100, 10, 0, 100
25	$(13, 7, 5)$	5	20102010010201010, 2010010, 2010, 201010, 2010201001020101020100 10
25	$(14, 8, 3)$	7	100102010010201010201001020102, 010102, 0, 0100102, 100102, 1001 02010010201010201001020102010010201010201001020102, 010102010 010201010201001020102

Table C.2: *Semi-Abelian returns in the Tribonacci word*

C.3 4-bonacci word

n	Ab. class	#sAr	semi-Abelian returns (sAr)
1	$(1,0,0,0)$	4	0, 01, 02, 03
1	$(0,1,0,0)$	4	1020, 10, 1030, 100
1	$(0,0,1,0)$	4	2010010, 20103010, 2010, 201010
1	$(0,0,0,1)$	4	301020102010, 301020100102010, 30102010, 30102010102010
2	$(1,1,0,0)$	5	1, 10, 0, 103, 102
2	$(1,0,1,0)$	5	2010301, 0, 20101, 201, 201001
2	$(1,0,0,1)$	5	3010201, 30102010201, 0, 30102010010201, 3010201010201
2	$(2,0,0,0)$	4	0010201030102010102010301020, 001020103010201020103010201, 00 102010301020103010201, 001020103010201
3	$(2,1,0,0)$	5	1, 0, 0103, 01, 0102
3	$(1,1,1,0)$	5	20, 20100, 10, 2010, 201030
3	$(2,0,1,0)$	4	0201, 020101, 02010301, 0201001
3	$(1,1,0,1)$	5	301020, 3010201020, 301020101020, 10, 3010201001020
3	$(2,0,0,1)$	4	03010201, 030102010010201, 03010201010201, 030102010201
3	$(1,2,0,0)$	4	10102010301020100102010301020, 101020103010201001020103010201 0301020100102010301020, 1010201030102010010201030102010010201 0301020, 1010201030102010010201030102010201030102010010201030 1020
4	$(2,1,1,0)$	6	2, 1, 0, 20103, 201, 2010
4	$(2,1,0,1)$	6	1, 0, 301020102, 30102010102, 301020100102, 30102
4	$(3,1,0,0)$	5	0010201030102, 001020103010201010201030102, 001020103010201030 102, 01, 0010201030102010201030102
4	$(2,2,0,0)$	6	10010201030102, 0, 100102010301020102010301020, 100102010301020 10301020, 10102010301020, 100102010301020
5	$(3,1,1,0)$	6	0201, 020103, 0, 201, 01, 02
5	$(2,2,1,0)$	5	1020, 20, 10, 1020100, 10201030
5	$(2,1,1,1)$	5	3010, 30102010, 30102010010, 2010, 3010201010
5	$(3,1,0,1)$	5	030102, 0301020102, 0301020100102, 01, 030102010102
5	$(2,2,0,1)$	4	103010201020, 103010201001020, 10301020, 10301020101020
5	$(3,2,0,0)$	5	10010201030102, 1001020103010201030102, 0, 01010201030102, 10010 201030102010201030102
5	$(2,1,2,0)$	4	20102010301020100102010301020101020103010201001020103010, 201 020103010201001020103010201010201030102010010201030102010301 020100102010301020101020103010201001020103010, 20102010301020 100102010301020101020103010201001020103010201010201030102010 01020103010, 201020103010201001020103010201010201030102010010 2010301020100102010301020101020103010201001020103010
6	$(3,2,1,0)$	7	2, 20, 1020103, 1, 10, 0, 102
6	$(3,1,1,1)$	6	3010201, 0, 301020101, 301, 201, 3010201001
6	$(3,2,0,1)$	5	1030102010102, 10301020100102, 0, 10301020102, 1030102
6	$(4,1,1,0)$	5	0201, 0010201030102010301, 00102010301, 00102010301020102010301, 0010201030102010102010301

Table C.3: *Semi-Abelian returns in the 4-bonacci word*

n	Ab. class	#sAr	semi-Abelian returns (sAr)
6	$(4, 2, 0, 0)$	4	01001020103010201010201030102, 01001020103010201030102, 010010201030102010201030102, 010010201030102
6	$(3, 1, 2, 0)$	5	0, 2010201030102010010201030102010102010301020100102010301, 20102010301020100102010301020101020103010201001020103010201030102010010201030102010102010301020100102010301, 20102010301020100102010301020101020103010201001020103010201001020103010201010201030102010010201030102010301, 201020103010201001020103010201010201030102010010201030102010102010301020100102010301
7	$(4, 2, 1, 0)$	7	2, 1, 0, 01020103, 0102, 01, 02
7	$(3, 2, 1, 1)$	6	301020, 20, 10, 30, 30102010, 301020100
7	$(4, 1, 1, 1)$	5	0301020101, 0201, 0301, 03010201, 03010201001
7	$(4, 2, 0, 1)$	4	010301020100102, 01030102010102, 01030102, 010301020102
7	$(3, 3, 1, 0)$	5	1020, 101020103010201001020103010201020103010201001020103 0, 101020103010201001020103 0, 101020103010201001020103010201030102010010201030, 10102010301020100102010301020100102010 30
7	$(3, 2, 2, 0)$	6	201010201030102010010201030 10, 10, 2010102010301020100102010301020103010201001020103010, 201020103010201001020103010, 201010201030102010010201030, 2010102010301020100102010301020100 1020103010
7	$(4, 1, 2, 0)$	4	0201020103010201001020103010201010201030102010010201030102010010201030102010102010301020100102010301, 02010201030102010010201030102010102010301020100102010301, 020102010301020100102010301020101020103010201001020103010201010201030102010010201 0301, 02010201030102010010201030102010102010301020100102010301020103010201001020103010201010201030102010010 2010301
8	$(4, 2, 1, 1)$	7	3, 3010201, 2, 1, 0, 30102010, 30102
8	$(5, 2, 1, 0)$	6	00102010301020102010 3, 001020103010 20103, 001020103, 00102010301020101020103, 01, 02
8	$(4, 3, 1, 0)$	8	1020, 0, 10010201030, 1001020103010201030, 1010201030, 102, 1001020103, 10010201030102010201030
8	$(4, 2, 2, 0)$	8	201001020103, 20100102010301, 2010010201030102010 3010, 1, 201020103010, 0, 20101020103010, 201001020103010
9	$(5, 2, 1, 1)$	7	030102, 3010201, 0, 03010201, 01, 02, 03
9	$(4, 3, 1, 1)$	6	301020, 1020, 1030, 10, 10301020100, 10301020
9	$(4, 2, 2, 1)$	5	201030102010010, 3010, 20103010, 2010, 20103010201010
9	$(5, 3, 1, 0)$	7	0, 10010201030102010 3, 1001020103010201020103, 0101020103, 1001020103, 102, 0102
9	$(5, 2, 2, 0)$	7	020102010301, 201001020103, 20100102010301, 0, 2010010201030102010301, 01, 02010102010301
9	$(4, 3, 2, 0)$	5	10, 102010201030102010010201030, 201010201030102010010201030, 201010201030102010010201030102010010201030, 2010102010301020100102010301020103010201001020 1030

Table C.3: Semi-Abelian returns in the 4-bonacci word

n	Ab. class	#sAr	semi-Abelian returns (sAr)
9	$(4, 2, 1, 2)$	4	301020103010201001020103010201010201030102010010201030102010201030102010010201030102010102010301020100102010301020100102010301020101020103010201001020103010201020103010201001020103010201010201030102010010201, 301020103010201001020103010201010201030102010010201030102010201020103010201001020103010201020103010201001020103010201010201030102010010201, 301020103010201001020103010201010201030102010010201030102010201020103010201001020103010201010201030102010010201030102010201020103010201001020103010201010201030102010010201, 301020103010201001020103010201010201030102010010201030102010201030102010010201030102010102010301020100102010301020101020103010201001020103010201020103010201001020103010201010201030102010010201
10	$(5, 3, 1, 1)$	8	301020, 1, 10, 0, 103, 1030102, 30102, 102
10	$(5, 2, 2, 1)$	6	2010301, 20103010201001, 0, 301, 201, 2010301020101
10	$(6, 2, 1, 1)$	5	0010201, 0010201030102010102 01, 03010201, 001020103010201, 0010201030102010201
10	$(6, 3, 1, 0)$	5	0100102010301020103, 01001020103, 0100102010301020102 0103, 0102, 01001020103010201010 20103
10	$(5, 3, 2, 0)$	8	201001020103, 1, 10, 201010201030, 0, 2010010201030102010 30, 102010201030, 2010010201030
10	$(6, 2, 2, 0)$	4	020100102010301020102010301, 020100102010301, 0201001020103010 2010301, 0201001020103010201010 2010301
10	$(5, 2, 1, 2)$	5	0, 301020103010201001020103010201010201030102010010201030102010201030102010010201030102010102010301020100 10201, 30102010301020100102010301020101020103010201001020103010201020103010201001020103010201010201030102010010201, 301020103010201001020103010201010201030102010010201030102010201020103010201001020103010201020103010201001020103010201010201030102010010201, 301020103010201001020103010201010201030102010010201030102010201030102010010201030102010102010301020100102010301020100102010301020101020103010201001020103010201020103010201001020103010201010201030102010010201

Table C.3: Semi-Abelian returns in the 4-bonacci word

C.4 7-bonacci word

n	Ab. class	#sAr	semi-Abelian returns (sAr)
1	$(1,0,0,0,0,0,0)$	7	0, 04, 05, 06, 01, 02, 03
1	$(0,1,0,0,0,0,0)$	7	1050, 1020, 10, 1030, 1040, 100, 1060
1	$(0,0,1,0,0,0,0)$	7	20104010, 2010010, 20106010, 20103010, 20105010, 2010, 201010
1	$(0,0,0,1,0,0,0)$	7	3010201050102010, 3010201040102010, 301020102010, 301020100102010, 30102010, 3010201060102010, 30102010102010
1	$(0,0,0,0,1,0,0)$	7	4010201030102010201030102010, 401020103010201030102010, 401020103010201010201030102010, 4010201030102010010201030102010, 40102010301020105010201030102010, 40102010301020106010201030102010, 4010201030102010
1	$(0,0,0,0,0,1,0)$	7	50102010301020104010201030102010, 50102010301020104010201030102010301020104010201030102010, 501020103010201040102010301020100102010301020104010201030102010, 5010201030102010401020103010201060102010301020104010201030102010, 50102010301020104010201030102010102010301020104010201030102010, 501020103010201040102010301020102010301020104010201030102010, 50102010301020104010201030102010401020103010 2010
1	$(0,0,0,0,0,0,1)$	7	601020103010201040102010301020105010201030102010401020103010201050102010301020104010201030102010, 60102010301020104010201030102010501020103010201040102010301020102010301020104010201030102010501020103010201040102010301 02010, 601020103010201040102010301020105010201030102010401020103010201030102010401020103010201050102010301020104010201030102010, 60102010301020104010201030102010501020103010201040102010301020101020103010201040102010301020105010201030102010401020103010 2010, 6010201030102010401020103010201050102010301020104010201030102010401020103010201050102010301020104010201030102010, 601020103010201040102010301020105010201030102010401020103010 2010, 601020103010201040102010301020105010201030102010401020103010201001020103010201040102010301020105010201030102010401020103010 2010
2	$(1,1,0,0,0,0,0)$	8	1, 10, 0, 106, 105, 104, 103, 102
2	$(1,0,1,0,0,0,0)$	8	2010301, 2010401, 0, 2010501, 20101, 201, 2010601, 201001
2	$(1,0,0,1,0,0,0)$	8	301020106010201, 3010201, 30102010201, 0, 301020104010201, 30102010010201, 301020105010201, 3010201010201
2	$(1,0,0,0,1,0,0)$	8	401020103010201, 401020103010201020103010201, 0, 4010201030102010601020103010201, 40102010301020101020103010201, 401020103010201001020103010201, 4010201030102010501020103010201, 40102010301020103010201

__Table C.4:__ Semi-Abelian returns in the 7-bonacci word

n	Ab. class	#sAr	semi-Abelian returns (sAr)
2	$(1,0,0,0,0,1,0)$	8	5010201030102010401020103010201, 50102010301020104010201030102010401020103010201, 0, 5010201030102010401020103010201010201030102010401020103010201, 501020103010201040102010301020102010301020104010201030 10201, 50102010301020104010201030102010301020104010201030 10201, 501020103010201040102010301020100102010301020104010201030 10201, 5010201030102010401020103010201060102010301020104010201030 10201
2	$(1,0,0,0,0,0,1)$	8	601020103010201040102010301020105010201030102010401020103010201010201030102010401020103010201050102010301020104010201030 10201, 601020103010201040102010301020105010201030102010401020103010201001020103010201040102010301020105010201030102010401020103010201, 0, 601020103010201040102010301020105010201030102010401020103010201, 60102010301020104010201030102010501020103010201040102010301020102010301020104010201030102010501020103010201040102010301 0201, 601020103010201040102010301020105010201030102010401020103010201040102010301020105010201030102010401020103010201, 601020103010201040102010301020105010201030102010401020103010201030102010401020103010201050102010301020104 01020103010201, 60102010301020104010201030102010501020103010201040102010301020105010201030102010401020103010201

Table C.4: Semi-Abelian returns in the 7-bonacci word

n	Ab. class	#sAr	semi-Abelian returns (sAr)
2	$(2,0,0,0,0,0,0)$	7	001020103010201040102010301020105010201030102010401020103010 201060102010301020104010201030102010501020103010201040102010 301020105010201030102010401020103010201060102010301020104010 2010301020105010201030102010401020103010201, 0010201030102010 401020103010201050102010301020104010201030102010601020103010 201040102010301020105010201030102010401020103010201, 00102010 301020104010201030102010501020103010201040102010301020106010 201030102010401020103010201050102010301020104010201030102010 301020104010201030102010501020103010201040102010301020106010 20103010201040102010301020105010201030102010401020103010201, 001020103010201040102010301020105010201030102010401020103010 201060102010301020104010201030102010501020103010201040102010 301020104010201030102010501020103010201040102010301020106010 20103010201040102010301020105010201030102010401020103010201, 001020103010201040102010301020105010201030102010401020103010 201060102010301020104010201030102010501020103010201040102010 301020102010301020104010201030102010501020103010201040102010 301020106010201030102010401020103010201050102010301020104010 20103010201, 001020103010201040102010301020105010201030102010 401020103010201060102010301020104010201030102010501020103010 201040102010301020101020103010201040102010301020105010201030 102010401020103010201060102010301020104010201030102010501020 1030102010401020103010201, 0010201030102010401020103010201050 102010301020104010201030102010601020103010201040102010301020 105010201030102010401020103010201060102010301020104010201030 1020105010201030102010401020103010201
3	$(2,1,0,0,0,0,0)$	8	1, 0, 0105, 0106, 0103, 0104, 01, 0102
3	$(1,1,1,0,0,0,0)$	8	20, 20100, 10, 201060, 2010, 201040, 201030, 201050
3	$(2,0,1,0,0,0,0)$	7	0201, 020101, 02010601, 02010501, 02010401, 02010301, 0201001
3	$(1,1,0,1,0,0,0)$	8	301020, 3010201020, 301020101020, 10, 30102010401020, 30102010501 020, 3010201001020, 30102010601020
3	$(2,0,0,1,0,0,0)$	7	03010201, 030102010010201, 0301020105010201, 03010201010201, 030 1020106010201, 0301020104010201, 030102010201
3	$(1,1,0,0,1,0,0)$	8	40102010301020, 10, 4010201030102010102010301020, 4010201030102 010301020, 40102010301020102010301020, 40102010301020105010201 0301020, 40102010301020100102010301020, 4010201030102010601020 10301020
3	$(2,0,0,0,1,0,0)$	7	04010201030102010501020103010201, 040102010301020102010301020 1, 040102010301020101020103010201, 040102010301020106010201030 10201, 0401020103010201, 0401020103010201001020103010201, 04010 2010301020103010201

Table C.4: Semi-Abelian returns in the 7-bonacci word

n	Ab. class	#sAr	semi-Abelian returns (sAr)
3	$(1,1,0,0,0,1,0)$	8	10, 5010201030102010401020103010201001020103010201040102010301020, 50102010301020104010201030102010601020103010201040102010301020, 501020103010201040102010301020102010301020104010201030102 0, 501020103010201040102010301020, 501020103010201040102010301020104010201030102 0, 501020103010201040102010301020101020103010201040102010301020, 50102010301020104010201030102010301020104010201030102 0
3	$(2,0,0,0,0,1,0)$	7	05010201030102010401020103010201030102010401020103010201, 050102010301020104010201030102010201030102010401020103010201, 05010201030102010401020103010201010201030102010401020103010201, 05010201030102010401020103010201060102010301020104010201030102 01, 05010201030102010401020103010201040102010301 0201, 0501020103010201040102010301 0201, 050102010301020104010201030102010010201030102010401020103 0201
3	$(1,1,0,0,0,0,1)$	8	6010201030102010401020103010201050102010301020104010201030102010301020104010201030102010501020103010201040102010301020, 10, 6010201030102010401020103010201050102010301020104010201030 1020, 601020103010201040102010301020105010201030102010401020103010201040102010301020105010201030102010401020103 01020, 601020103010201040102010301020105010201030102010401020103010201020103010201040102010301020105010201030102010401020103 01020, 6010201030102010401020103010201050102010301020104010201030102010102010301020104010201030102010501020103010201040102010301020, 6010201030102010401020103010201050102010301020104010201030102010010201030102010401020103010201050102010301020104010201030 1020, 601020103010201040102010301020105010201030102010401020103010201050102010301020104010201030 1020
3	$(2,0,0,0,0,0,1)$	7	0601020103010201040102010301020105010201030102010401020103010201030102010401020103010201050102010301020104010201030 10201, 0601020103010201040102010301020105010201030102010401020103010201010201030102010401020103010201050102010301020104010201030 10201, 06010201030102010401020103010201050102010301020104010201030102010010201030102010401020103010201050102010301020104010201030 10201, 06010201030102010401020103010201050102010301020104010201030102 01, 0601020103010201040102010301020105010201030102010401020103010201, 0601020103010201040102010301020105010201030102010401020103010201020103010201040102010301020105010201030102010401020103010201, 06010201030102010401020103010201050102010301020104010201030102010401020103010201050102010301020104010201030 10201

Table C.4: Semi-Abelian returns in the 7-bonacci word

n	**Ab. class**	**#sAr**	**semi-Abelian returns (sAr)**
3	$(1,2,0,0,0,0,0)$	7	101020103010201040102010301020105010201030102010401020103010 201060102010301020104010201030102010501020103010201040102010 301020100102010301020104010201030102010501020103010201040102 010301020106010201030102010401020103010201050102010301020104 010201030102010401020103010201050102010301020104010201030102 010601020103010201040102010301020105010201030102010401020103 010201001020103010201040102010301020105010201030102010401020 103010201060102010301020104010201030102010501020103010201040 102010301020, 10102010301020104010201030102010501020103010201 040102010301020106010201030102010401020103010201050102010301 020104010201030102010010201030102010401020103010201050102010 301020104010201030102010601020103010201040102010301020105010 201030102010401020103010201060102010301020104010201030102010 501020103010201040102010301020100102010301020104010201030102 010501020103010201040102010301020106010201030102010401020103 0102010501020103010201040102010301020, 1010201030102010401020 103010201050102010301020104010201030102010601020103010201040 102010301020105010201030102010401020103010201001020103010201 040102010301020105010201030102010401020103010201060102010301 020104010201030102010501020103010201040102010301020, 10102010 301020104010201030102010501020103010201040102010301020106010 201030102010401020103010201050102010301020104010201030102010 010201030102010401020103010201050102010301020104010201030102 010601020103010201040102010301020105010201030102010401020103 010201001020103010201040102010301020105010201030102010401020 103010201060102010301020104010201030102010501020103010201040 102010301020, 10102010301020104010201030102010501020103010201 040102010301020106010201030102010401020103010201050102010301 020104010201030102010010201030102010401020103010201050102010 301020104010201030102010601020103010201040102010301020105010 201030102010401020103010201030102010401020103010201050102010 301020104010201030102010601020103010201040102010301020105010 201030102010401020103010201001020103010201040102010301020105 010201030102010401020103010201060102010301020104010201030102 0105010201030102010401020103010201 0, 10102010301020104010201030 102010501020103010201040102010301020106010201030102010401020 103010201050102010301020104010201030102010010201030102010401 020103010201050102010301020104010201030102010601020103010201 040102010301020105010201030102010401020103010201020103010201 040102010301020105010201030102010401020103010201060102010301 020104010201030102010501020103010201040102010301020100102010 301020104010201030102010501020103010201040102010301020106010 2010301020104010201030102010501020103010201040102010301020, 1 010201030102010401020103010201050102010301020104010201030102 010601020103010201040102010301020105010201030102010401020103 010201001020103010201040102010301020105010201030102010401020 103010201060102010301020104010201030102010501020103010201040 102010301020105010201030102010401020103010201060102010301020 104010201030102010501020103010201040102010301020100102010301 020104010201030102010501020103010201040102010301020106010201 0301020104010201030102010501020103010201040102010301020

Table C.4: *Semi-Abelian returns in the 7-bonacci word*

n	Ab. class	#sAr	semi-Abelian returns (sAr)
4	$(2,1,1,0,0,0,0)$	9	2, 1, 0, 20103, 20104, 201, 2010, 20105, 20106
4	$(2,1,0,1,0,0,0)$	9	3010201040102, 1, 0, 301020102, 30102010102, 3010201050102, 301020100102, 3010201060102, 30102
4	$(2,1,0,0,1,0,0)$	9	1, 4010201030102010010201030102, 4010201030102010501020103010 2, 0, 401020103010201030102, 4010201030102, 4010201030102010201030102, 4010201030102010601020103010 2, 401020103010201010201030102
4	$(2,1,0,0,0,1,0)$	9	501020103010201040102010301020104010201030102, 1, 0, 5010201030102010401020103010201030102010401020103 0102, 501020103010201040102010301020102010301020104010201030102, 50102010301020104010201030102, 50102010301020104010201030102010102010301020104010201030102, 50102010301020104010201030102010010201030102010401020103 0102, 5010201030102010401020103010201060102010301020104010201030102
4	$(2,1,0,0,0,0,1)$	9	60102010301020104010201030102010501020103010201040102010301020100102010301020104010201030102010501020103010201040102010 30102, 1, 0, 601020103010201040102010301020105010201030102010401 0201030102, 601020103010201040102010301020105010201030102010401020103010201050102010301020104010201030102, 60102010301020104010201030102010501020103010201040102010301020103010201040102010301020105010201030102010401020103 0102, 601020103010201040102010301020105010201030102010401020103010201040102010301020105010201030102010401020103 0102, 601020103010201040102010301020105010201030102010401020103010201020103010201040102010301020105010201030102010401020103 0102, 6010201030102010401020103010201050102010301020104010201030102010102010301020104010201030102010501020103010201040102010 30102

Table C.4: Semi-Abelian returns in the 7-bonacci word

n	**Ab. class**	**#sAr**	**semi-Abelian returns (sAr)**
4	$(3, 1, 0, 0, 0, 0, 0)$	8	001020103010201040102010301020105010201030102010401020103010 201060102010301020104010201030102010501020103010201040102010 301020103010201040102010301020105010201030102010401020103010 201060102010301020104010201030102010501020103010201040102010 30102, 001020103010201040102010301020105010201030102010401020 103010201060102010301020104010201030102010501020103010201040 10201030102, 001020103010201040102010301020105010201030102010 401020103010201060102010301020104010201030102010501020103010 201040102010301020101020103010201040102010301020105010201030 102010401020103010201060102010301020104010201030102010501020 10301020104010201030102, 001020103010201040102010301020105010 201030102010401020103010201060102010301020104010201030102010 501020103010201040102010301020104010201030102010501020103010 201040102010301020106010201030102010401020103010201050102010 301020104010201030102, 00102010301020104010201030102010501020 103010201040102010301020106010201030102010401020103010201050 102010301020104010201030102010601020103010201040102010301020 1050102010301020104010201030102, 0010201030102010401020103010 201050102010301020104010201030102010601020103010201040102010 301020105010201030102010401020103010201050102010301020104010 201030102010601020103010201040102010301020105010201030102010 4010201030102, 01, 0010201030102010401020103010201050102010301 020104010201030102010601020103010201040102010301020105010201 030102010401020103010201020103010201040102010301020105010201 030102010401020103010201060102010301020104010201030102010501 02010301020104010201030102

Table C.4: *Semi-Abelian returns in the 7-bonacci word*

n	Ab. class	#sAr	semi-Abelian returns (sAr)
4	$(2, 2, 0, 0, 0, 0, 0)$	9	1001020103010201040102010301020105010201030102010401020103010201060102010301020104010201030102010501020103010201040102010301020105010201030102010401020103010201060102010301020104010201030102010501020103010201040102010301020, 10010201030102010401020103010201050102010301020104010201030102010601020103010201040102010301020105010201030102010401020103010201020103010201040102010301020105010201030102010401020103010201060102010301020104010201030102010501020103010201040102010301020, 0, 10102010301020104010201030102010501020103010201040102010301020106010201030102010401020103010201050102010301020104010201030 1020, 1001020103010201040102010301020105010201030102010401020103010201060102010301020104010201030102010501020103010201040102010301020106010201030102010401020103010201050102010301020104010201030 1020, 1001020103010201040102010301020105010201030102010401020103010201060102010301020104010201030102010501020103010201040102010301 02, 10010201030102010401020103010201050102010301020104010201030102010601020103010201040102010301020105010201030102010401020103010 20, 10010201030102010401020103010201050102010301020104010201030102010601020103010201040102010301020105010201030102010401020103010201040102010301020105010201030102010401020103010201060102010301020104010201030102010501020103010201040102010301020, 1001020103010201040102010301020105010201030102010401020103010201060102010301020104010201030102010501020103010201040102010301020103010201040102010301020105010201030102010401020103010201060102010301020104010201030102010501020103010201040102010301020

Table C.4: Semi-Abelian returns in the 7-bonacci word